LONG-TIME EXISTENCE FOR THE CALABI FLOW ON RULED MANIFOLDS OVER RIEMANN SURFACES

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ABSTRACT. Let $(E, h) \to (\Sigma, \omega_\Sigma)$ Hermitian vector bundle equipped with a holomorphic structure $\overline{\partial}_E$ determining a holomorphic vector bundle $\mathcal{E}$, where the base Σ is a Riemann surface, and ω_Σ is a Kähler metric of constant scalar curvature. Consider the projectivisation $\mathbb{P}(\mathcal{E})$. We write J for complex structure on this manifold, and $\omega_k(h, J)$ for the adiabatic Kähler metrics determined by

$$F_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} + ik\pi^*\omega_\Sigma,$$

where $F_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ is the curvature of the hyperplane line bundle over $\mathbb{P}(\mathcal{E})$, π is the natural map from the projectivisation, and $k >> 0$. Then if $\mathcal{E}$ is simple and moreover satisfies a natural condition on its Harder-Narasimhan filtration, we prove the longtime existence of the Calabi flow starting from $\omega_k(h, J)$, verifying a conjecture of Chen in this special case.

1. INTRODUCTION

A central problem in Kähler geometry is to construct metrics of constant scalar curvature (cscK metrics) within a fixed Kähler class $[\omega]$. More particularly, one of the main aims of the field is to characterise the existence of such metrics by an algebraic geometric stability condition. This problem was solved in the special case of Kähler-Einstein metrics by Chen-Donaldson-Sun in [CDS], and is completely analogous to the more classical Kobayashi-Hitchin correspondence, proven in the 1980s by Donaldson, and Uhlenbeck-Yau (see [DO1], [DO2], and [UY]). The latter result, also known as the Donaldson-Uhlenbeck-Yau (DUY) theorem, characterises the existence of Hermitian-Einstein metrics, or equivalently Hermitian-Yang-Mills (HYM) connections; metrics (connections) whose contracted curvature is a constant multiple of the identity, on a holomorphic vector bundle over a Kähler manifold. The general cscK problem remains open, and even the precise stability condition that should be required remains elusive, but there are known algebraic-geometric obstructions to existence.

Because these canonical metrics arise as the absolute minimisers of energy functionals on certain infinite dimensional spaces, one approach to the above problems is to consider their the gradient flows and try to prove their longtime existence and convergence to a minimiser, whenever a suitable algebraic-geometric condition is met. Due to the infinite dimensionality of the spaces in question, this is in general a difficult problem, but in the case of HYM connections, this idea was successfully carried out by Donaldson in [DO1] and [DO2] (at least in the projective setting), where the correct condition on the bundle is the classical Mumford-Takemato slope stability. The gradient flow is known as the Yang-Mills flow. More generally, as discussed below, even in the case of an unstable bundle, the longtime existence and convergence of this flow is in some sense completely understood on a general Kähler manifold. The gradient flow designed to find cscK metrics when they exist is known as the Calabi flow. This is a fourth order parabolic equation for a path of Kähler metrics,

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and in contrast to the Yang-Mills flow, relatively little is known about it in complex dimensions >1. It is much more difficult to show that it achieves its goal of finding the cscK metric when one exists, before even considering the study of the longtime existence and asymptotic properties in general. Indeed, when the complex dimension is at least 2, there are very few examples of complex manifolds such that the flow starting from a given Kähler metric has been shown to exist for all time. The purpose of the present paper is to rectify this situation somewhat by attacking the long-time existence problem in the special case of a projective vector bundle over a Riemann surface, where we may exploit the existence and long-time behaviour of Yang-Mills flow.

1.1. **The gradient flows in general.** In both of the problems discussed above there are two different points of view one may take. In the cscK case, one may fix a complex manifold (X, J) and look for a compatible cscK metric g_J (or Kähler form ω_J), or fix a symplectic form ω and search for a compatible integrable almost complex structure $J(\omega)$ so that the resulting metric $g(\omega, J(\omega))$ is cscK. Similarly, on a complex vector bundle E over a fixed Kähler manifold (X, ω), one may fix either the holomorphic structure (equivalently d-bar operator) $\mathcal{E} = (E, \overline{\partial}_E)$ or the hermitian metric h, and vary the other structure, with the aim of finding either an Hermitian-Einstein metric $h_{\mathcal{E}}$, or an HYM holomorphic structure (equivalently Chern connection) $A = (\overline{\partial}_E, h)$.

This duality gives rise to two parallel variational problems corresponding to energy functionals on two different spaces. Namely, in the bundle setting, the most natural functional to consider for a fixed Hermitian bundle (E, h) is the Yang-Mills energy

$$
\begin{aligned}
YM \quad &: \quad \mathcal{A}_h^{1,1}(E) \to \mathbb{R} \\
A \quad &\to \quad \int_X |F_A|^2 \, dvol_{g(\omega)}
\end{aligned}
\tag{1.1}
$$

where $\mathcal{A}_h^{1,1}(E)$ is the space of integrable, metric connections. In the cscK problem, if we fix a Kähler manifold (X, J_0, g, ω) and write $\mathcal{J}^{int}(X, \omega)$ for the space of ω-compatible integrable almost complex structures, then the natural functional is the Calabi energy:

$$
\begin{aligned}
C \quad &: \quad \mathcal{J}^{int}(X, \omega) \to \mathbb{R} \\
J \quad &\to \quad \int_X (Scal(J))^2 \, dvol_g,
\end{aligned}
\tag{1.2}
$$

where $Scal(J)$ is the scalar curvature of the metric associated to ω and J. Then HYM connections and cscK metrics respectively arise as the absolute minimisers of these functionals.

The negative gradient flows of these functionals are given by

$$
\frac{\partial A_t}{\partial t} = -d_{A_t}^* F_{A_t} \tag{1.3}
$$

and

$$
\frac{dJ_t}{dt} = -\frac{1}{2} J_t \mathfrak{D}_{J_t} Scal(J_t)), \tag{1.4}
$$

or equivalently

$$
\frac{dJ_t}{dt} = \frac{1}{2} \mathcal{L}_{Re\nabla^{1,0} Scal(J_t)} J_t.
$$

Equation 1.3 is known as the **Yang-Mills flow**. Equation 1.4 is implicit in the work of Donaldson (see [DO3]), but to the author's knowledge, it first appeared explicitly in the paper [CS] by Chen and Sun, and so we will refer to it as the **Chen-Sun flow**.

It is convenient for many purposes (again as in [DO1] and [DO2]), to take the alternative point of view and consider certain functionals on the space of Hermitian metrics $\mathrm{Herm}^+(E)$ and Kähler potentials $\mathcal{H}_\omega$ such that the Hermitian-Einstein metrics and cscK metrics are global minimisers. These two functionals called the Donaldson functional and the Mabuchi energy respectively, are slightly more difficult to describe, but their gradient flows are easy to write down and are given by:

$$h_t^{-1}\frac{\partial h_t}{\partial t} = -2\left(\Lambda_\omega F_{(\overline{\partial}_E, h_t)} - \mu_\omega(E) Id_E\right) \tag{1.5}$$

and

$$\frac{\partial \omega_t}{\partial t} = -i\overline{\partial}_J \partial_J Scal(\omega_t), \tag{1.6}$$

where $\mu_\omega(E)$ is the slope of the bundle E and $\omega_t = \omega + i\overline{\partial}_J\partial_J\phi_t$, for a some path of Kähler potentials ϕ_t. Equation 1.5 is called the **Hermitian-Yang-Mills flow** (or **Donaldson heat flow**), and was employed by Donaldson in [DO1] to prove the long-time existence of Equation 1.3, on a general Kähler manifold, a critical first step in the proof of the DUY theorem. Equation 1.6 is built to find CSK metrics, and is the main object of study in this paper. It is called the **Calabi flow**. It's clear that the fixed points of these flows are precisely the Hermitian-Einstein and cscK metrics respecitively.

In each of these problems, there is a natural groups of symmetries. In the HYM problem we may consider the group $\mathcal{G}(E, h)$, of unitary gauge transformations, which are smooth isomorphisms of E preserving the metric h. In the cscK case one may consider $\mathcal{G}(X, \omega)$, the group of Hamiltonian symplectomorphisms of the symplectic manifold (X, ω). The actions of $\mathcal{G}(E, h)$ and $\mathcal{G}(X, \omega)$ preserve the spaces $\mathcal{A}_h^{1,1}(E)$ and $\mathcal{J}^{int}(X, \omega)$, as well as the two flows 1.3 and 1.4. The two functionals defined above descend to the quotients by these actions.

Equations 1.3 and 1.4 and Equations 1.5 and 1.6 are equivalent in the sense that given solutions to the former one may construct solutions to the latter in a natural way, and the converse is true up to the action of the of the groups $\mathcal{G}(E, h)$ and $\mathcal{G}(X, \omega)$ (see either [DO1] Section 1.1, [DOKR] Section 6.3.1, or also Section 3.4 below for the Yang-Mills flow, and [CS] Lemma 5.1 for Calabi flow). Note that equations 1.5 and 1.6 are parabolic, whereas equations 1.3 and 1.4 are not due to the invariance under the symmetry groups. The advantage of the first two equations however, is that whereas the HYM flow and the Calabi flow must blow up in infinite time in the case that no canonical metric exists, their analogues with moving holomorphic structure may still converge in the absence of such a fixed point.

Indeed, for the Yang-Mills flow this is a well-studied problem. Using deep gauge theoretic results of Uhlenbeck see [U1] and [U2] one may see easily that on a general Kähler manifold, a subsequence along this flow has a limit in a certain generalised sense. The limiting connection can be singular in complex dimensions ≥ 3, and can live on a different topological bundle if $\dim_{\mathbb{C}} = 2$. Moreover the convergence must take "bubbling" phenomena into account. When the base is a Riemann surface however, these phenomena do not appear, and the convergence is in the usual C^∞ sense. Going further Daskalapoulos [D] proved that in fact the limiting connection is independent of the subsequence chosen, and the flow converges to a connection determined by a certain canonical algebraic-geometric object derived from the Harder-Narasimhan filtration of the initial holomorphic bundle $\mathcal{E}_0$ (see Theorem 3.5 below). In particular, the limit will in general merely be a critical point of the functional 1.1, a so-called Yang-Mills connection, when the bundle $\mathcal{E}_0$ is not stable, rather than a minimiser. The Yang-Mills connections are direct sums of Hermitian-Yang-Mills connections on direct summands with possibly different slopes. In general then, the holomorphic structure induced by this limiting connection will be different than that of the original bundle; this phenomenon is the well-known "jumping" of holomorphic structures. There are also generalisations of the result of Daskalapoulos to higher dimensions that deal with bubbling and the various singularities that occur (see [DW1], [DW2], [S], [SW]). In the event that the bundle admits a Yang-Mills connection already, these theorems imply the convergence of the flow to this connection. In this case the jumping phenomenon does not occur.

The theory of cscK metrics and the Calabi flow is far less developed, although conjecturally a similar sort of picture exists. In the first place, there is no analogue of the DUY theorem, and although one expects the existence problem to again be equivalent to some notion of stability (this is commonly known as the Donaldson-Tian-Yau conjecture), it remains unclear what the precise condition is, as K-stability, which is sufficient in the Kähler -Einstein problem (this is precisely the CDS theorem) is unlikely to be sufficient in general (see [ACGTF]). A good replacement candidate was given by Szekelyhidi in [SZ2]. Still more generally, one could also consider the analogue of the Yang-Mills connections in this setting, which are the critical points of the functional 1.2, namely, solutions of the equation

$$\mathcal{L}_{Re\nabla^{1,0}Scal(J)}J = 0. \tag{1.7}$$

The resulting Kähler metrics are known as **extremal metrics**.

Even less is known about the flow. Indeed even longtime existence is, in general unknown (see the discussion of Chen's conjecture below). Notice that the fixed points of the flow 1.4 are precisely the solutions to 1.7. When such a critical point in the isomorphism class of a complex structure J_0 exists, we expect it to be realised as the limit of the Chen-Sun flow. More generally, if any such holomorphic structure exists, then the Chen-Sun flow should converge to it starting from any J_0, where now the same jumping phenomenon as in the Yang-Mills case will occur. More precisely, we have the following conjectures.

Conjecture 1.1. *(Chen) The Calabi flow, starting from any Kähler metric exists for all time.*

Conjecture 1.2. *(see [DO3]) (Donaldson) Let (X, J_0, ω_0, g_0) be a Kähler manifold. Given a long-time solution ω_t to Calabi flow starting from ω_0 (inducing a solution J_t to equation 1.4) one of the following four conditions is satisfied:*

- *A cscK metric exists and Calabi flow converges to it.*
- *An extremal holomorphic structure J_∞ exists in the isomorphism class of J_0 and equation 1.4 converges to J_∞.*
- *An extremal holomorphic structure J_∞ exists in a different isomorphism class, and equation 1.4 converges to J_∞, giving rise to an extremal metric on on a different Kähler manifold with the same underlying smooth structure.*
- *The equation 1.4 converges to some sort of singular complex structure J_∞.*

The author has been unable to track down a precise reference for the first conjecture, but it is widely acknowledged to be due to X. Chen. Progress towards proofs of conjectures 1.1 and 1.2 has been slow so far. Both conjectures are known to be true in the case of Riemann surfaces (see [C], [Ch]), where it is clearly the first case of Conjecture 1.2 that is satisfied. For complex dimension greater than one, very few general results are known even about Chen's conjecture. The short-time existence of Calabi flow is known (see [CH]). The strongest results proven to date, also from [CH], are that the flow exists for all time and converges to a cscK metric when it is started sufficiently close to such a metric, and will exist for all time if the Ricci curvature remains bounded. Particular examples where Conjecture 1.1 is true may be found in [CH2], [FH], and [SZ]. To the author's knowledge this is essentially an exhaustive list of known example, and all of these exploit very particular symmetries of the geometries under consideration.

In the last three cases of Conjecture 1.2, the stated convergence should exhibit the failure of stability in a natural way, namely the limit should be the central fibre of a destabilising "test configuration". By analogy with the Yang-Mills setting, one might also expect the last case to give rise to a singular extremal metric in a sufficiently general sense. Some progress towards this conjecture has been made in [CSW] and [LWZ].

1.2. **Ruled manifolds over Riemann surfaces.** Let $(E, h) \to (\Sigma, \omega_\Sigma)$, be an Hermitian vector bundle where the base Σ is a Riemann surface, and ω_Σ is a metric of constant scalar curvature. If we fix a $\overline{\partial}_E$ operator on E, this determines a holomorphic vector bundle $\mathcal{E}$. If we write $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ for the hyperplane bundle over the projectivisation $\mathbb{P}(\mathcal{E})$, then for sufficiently large k, the two forms $F_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} + ik\pi^*\omega_\Sigma$, where $F_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)}$ is the curvature of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, equipped with the metric induced by h, determine Kähler metrics on $\mathbb{P}(\mathcal{E})$. In this paper we will study Conjecture 1.1 for the flow starting from these metrics.

1.3. **Adiabatic limits, and previous results.** The basic idea to prove long-time existence in this case is the notion of an adiabatic limit. This technique has been employed succesfully in a large range of geometric situations, and particularly in the construction of canonical Kähler metrics on various kinds of fibrations. One wants to solve an equation of the form $F(g) = 0$, and by stretching the base by a factor k (as we have done in our definition of $\omega_k(h, J)$ above), we produce a family of metrics and therefore obtain a family of equations of the form $F(g_k) = 0$. The adiabatic limit is the equation obtained by formally setting $k = \infty$, and may be thought of as approximation to the original equation for large k. With a solution to this equation in hand, we can in priniciple obtain a genuine solution to original equation for large enough k by using the implicit function theorem to perturb the adiabatic solution.

The first result in Kähler geometry along these lines was that of Hong [H]. He considers the fundamental question of when the manifold $\mathbb{P}(\mathcal{E})$ admits a cscK metric. Here the function F is the scalar curvature (and the metric g_k is our adiabatic metric), and by expanding the scalar curvature in powers of k^{-1} one may see that the adiabatic limit of this equation is precisely the Hermitian-Einstein equation. Hong's precise theorem is that for sufficiently large k the class $[\omega_k(h, J)]$ admits such a metric of constant scalar curvature if $\mathcal{E}$ is simple and admits an Hermitian-Einstein metric, and the base manifold Y (which in his case is allowed to be of arbitrary dimension), admits no holomorphic vector fields. By the DUY theorem, the hypotheses on $\mathcal{E}$ are exactly the hypothesis that $\mathcal{E}$ is slope stable. In a later paper [H3] Hong is able to relax the assumption on the simplicity of the bundle and the existence of vector fields on Y, by instead assuming the vanishing of a certain Futaki invariant.

A construction of cscK metrics on fibrations $X \to \Sigma$, where both Σ and the fibres of X are Riemann surfaces of genus ≥ 2 was given in [F] by Fine. It is based on the same geometric idea, although the details differ substantially, since the fibres of X admit moduli in this case.

Most relevant to the present paper is the article of Brönnle [B], which generalises [H], and considers the case $\mathbb{P}(\mathcal{E}) \to (Y, \omega_Y)$, where Y admits no holomorphic vector fields and ω_Y is a cscK metric. It is known by work or Ross and Thomas [RT] that if $\mathcal{E}$ is strictly unstable, then $\mathbb{P}(\mathcal{E})$ cannot admit a cscK metric. Brönnle's theorem is that when $\mathcal{E}$ splits as a direct sum

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m,$$

of stable bundles $\mathcal{E}_i$, all of which have different slopes $\mu_{\omega_Y}(\mathcal{E}_i)$, then the adiabatic classes admit *extremal* metrics. Note that the condition on $\mathcal{E}$ is just the condition that there is a Yang-Mills connection that gives a holomorphic structure in the isomorphism class of $\mathcal{E}$ (with the additional restriction on the equality of the slopes). Correspondingly, in this case the adiabatic limit is the general Yang-Mills equation

$$d_A^* F = 0.$$

In passing, we mention that a wide ranging generalisation of all of the above results is given by Dervan and Sektnan (see [DS1], [DS2] and [DS3]). They consider the case of a general holomorphic submersion $X \to Y$, where Y is a polarised manifold, and X has relatively ample line bundle. The adiabatic limit in this situation is a new equation, known as the optimal symplectic equation (respectively extremal symplectic equation), which generalises the notion of the HYM equation (respectively Yang-Mills equation).

1.4. **Overview of the proof.** The strategy of proof of all of the above theorems (see for example [F]), essentially follows the same trajectory, employing the adiabatic limit technique sketched above in a very precise way. Namely, the idea is to consider either the cscK or the extremal metrics as the zeros of a smooth map

$$F : V \to W$$

between two suitably defined Banach spaces. Then given a solution to the adiabatic limit equation, $\omega_k(h, J)$ will at least formally provide a solution to $F(g) = 0$ up to order k^{-2}. In order to obtain increasingly better approximations to this equation, one adds Kähler potentials to $\omega_k(h, J)$ in an attempt to eliminate the various terms of higher and higher orders in k^{-1}. This involves solving various linear elliptic pdes, the solutions of which are guaranteed by the geometry of the situation in question. One then shows that the approximation is in fact genuine, in the sense that it holds in the Banach space norm on W, rather than merely pointwise. Having established this, if one has suitable control on the linearisation dF_0, as well as on $F - dF_0$, then the quantitative version of the inverse function theorem will give an exact solution to the problem.

To solve Calabi flow, starting from $\omega_k(h, J)$ on the projectivisation $\mathbb{P}(\mathcal{E})$ of our bundle $(E, h) \to (\Sigma, \omega_\Sigma)$, where we have fixed some holomorphic structure $\overline{\partial}_E$, we would like to follow a similar strategy. However in all of the above scenarios, the equations under consideration are time independent, and in particular elliptic. As far as the authors are aware, the present work is the first example of a parabolic problem being solved in this way. The parabolic setting throws up several technical difficulties that do not occur with elliptic equations. First of all, for elliptic equations, the choice of the spaces V and W is more or less obvious, namely they will be Sobolev spaces with enough regularity to make things work. Once the geometry of the situation at hand is properly understood, the standard existence and elliptic regularity results may be applied, so that we obtain solutions to the required linear elliptic equations, with good estimates on the solutions in the chosen Banach spaces. Thus one may immediately conclude that a formal approximate solution is an approximate solution in the Banach space sense.

The first questions that one might naively ask are what the adiabatic limit of the Calabi flow is, and what the correct choice of V and W are for the problem at hand. It is relatively easy to see that in this case the adiabatic limit, as one might expect, is the HYM flow. That is, if we flow the metric h according to equation 1.5, then we obtain a path $\omega_k(h_t, J)$ Kähler forms that formally solves the equation

$$\frac{\partial \omega_k(h_t, J)}{\partial t} + i\overline{\partial}_J \partial_J Scal(\omega_k(h_t, J)) = k^{-2}\widehat{\sigma}_k(t), \tag{1.8}$$

where for each t we have a pointwise estimate

$$|\widehat{\sigma}_k(t)| \leq C,$$

where C is independent of k. We would like to define some sort of parabolic Sobolev spaces in which this inequality is true in the parabolic norm. There are several interlocking difficulties in doing this. First of all, one must to some extent develop the parabolic theory on compact manifolds. One source for this is the appendix of a paper by Huisken and Polden [HP]. In order to define a suitable norm that makes the parabolic theory work, a natural idea is to integrate the usual Sobolev norms (and the Sobolev norms of the derivatives in time) over the real line, but one has to build a certain weight function into the definition as well to insure the integrability of these quantities. In [HP] this is just an exponential function, because in their applications exponential convergence at infinity is guaranteed. One now encounters a problem, because we want to show that $\widehat{\sigma}_k(t)$ is actually bounded in this norm when k is large. However this quantity depends on the HYM flow, which blows up in infinite time unless the holomorphic bundle $\mathcal{E}$ is (poly)stable.

The way out is to change viewpoints, and consider the Yang-Mills flow, equation 1.3, instead (strictly speaking one must work with a sped up version). In other words we consider the path of metrics $\omega_k(h, J_t)$, where J_t is the holomorphic structure on the smooth manifold $\mathbb{P}(E)$ corresponding to the connection A_t on E along the flow. The flow A_t is determined entirely by a path of complex gauge transformations g_t; that is $A_t = g_t^*(A_0)$ for some $g_t \in \mathcal{G}^{\mathbb{C}}$, where $\mathcal{G}^{\mathbb{C}}$ is the group of smooth automorphisms of E. Since g_t also relates the Yang-Mills and Hermitian-Yang-Mills flow pictures, for the diffeomorphisms $\widetilde{g}_t$ of $\mathbb{P}(E)$ that they induce, we may write

$$\widetilde{g}_t^* \left(\omega_k(h, J_t)\right) = \omega_k(h_t, J).$$

The path of metrics $\omega_k(h, J_t)$ therefore gives a solution of equation 1.8 up to the diffeomorphisms $\widetilde{g}_t$, or more precisely, the equation

$$\frac{\partial \omega_k(h, J_t)}{\partial t} + i\overline{\partial}_{Jt}\partial_{J_t} Scal(\omega_k(h, J_t)) + \mathcal{L}_{V_t}\left(\omega_k(h, J_t)\right) = k^{-2}\sigma_k(t), \tag{1.9}$$

where V_t is the (time-dependent) generator of $\widetilde{g}_t$ and $\sigma_k(t) = (\widetilde{g}_t^{-1})^*(\widehat{\sigma}_k(t))$. Notice that this is now an equation on the moving complex manifold $(\mathbb{P}(E), J_t)$. The point here is that while g_t and therefore $\widetilde{g}_t$ fails to converge at infinity, destroying convergence of HYM flow, nevertheless the Yang-Mills flow itself converges (see Theorem 3.5 below), so the function $\sigma_k(t)$ will also converge, and there is hope of defining a parabolic norm such that this quantity is finite in the norm (and bounded in k).

However, here we encounter another issue, which is that (again unless the holomorphic structure defined by A_0 is polystable) the Yang-Mills flow does not converge exponentially, so we cannot use the analysis of [HP] as is. On the other hand, in [R], Råde, has shown that on a Riemann surface, for a general initial condition A_0 (inducing some arbitrary holomorphic bundle $\mathcal{E}_0$) the flow converges at a rate of $1/\sqrt{t}$ (again see Theorem 3.5). Then the first technical challenge is to find an appropriate weight function for the norm, so that $\|\sigma_k(t)\|$ is bounded, and at the same time the Lax-Milgram argument used to establish the linear parabolic existence and regularity theorems on compact manifolds in [HP] goes through.

Once this has been established, in a similar fashion to the elliptic versions of the problem, we may perturb the metrics $\omega_k(h, J_t)$ by adding paths of Kähler potentials to eliminate the higher order terms in equation 1.9. Writing out the effect of this on the scalar curvature, one sees that these potentials must satisfy various linear parabolic equations. By the argument described in the previous paragraph, we may find long-time solutions to these equations with estimates in the parabolic norm.

This means that we obtain metrics $\omega_{k,l}(t)$ solving the analogue of equation 1.9 where the right hand side instead has a factor of k^{-l} for l arbitrary, and moreover where the parabolic norm $\|\sigma_{k,l}(t)\|$ of the function it multiplies, is still bounded. Furthermore, using the convergence of Yang-Mills flow, and the linear parabolic theory, the metrics $\omega_{k,l}(t)$ will converge at infinity to Kahler metrics $\omega_{k,l,\infty}$.

The elliptic operators that appear in the parabolic equations we obtain are time dependent in some cases. In particular we obtain an equation involving the Laplacian on sections of E, depending on the connections A_t along the Yang-Mills flow. In order to apply the parabolic theory, we require that the right hand sides of our equations be orthogonal to the kernels of these operators, and also to those of their limits at infinity. For this we require two geometric hypotheses on the bundle. The former property is assured by the assumption of simplicity of the holomorphic bundle . The latter can be guaranteed if we know that the limit of the Yang-Mills flow at infinity is a Yang-Mills connection giving rise to a bundle that splits as a direct sum of stable bundles, all of which have different slope. By Theorem 3.5 below, this will happen precisely when another, fairly natural hypothesis on the initial holomorphic bundle $\mathcal{E} = \mathcal{E}_0$ is put in place, namely that its Harder-Narasimhan filtration is equal to any of its Harder-Narasimhan-Seshadri double filtrations; or in other words, the direct summands that appear in the associated graded object $Gr(\mathcal{E}_0)$ of the Harder-Narasimhan filtration are already stable. Notice that by the result of Theorem 3.5, as well as our assumption on $\mathcal{E}_0$, the metrics $\omega_{k,l,\infty}$ live on the manifold

$$(\mathbb{P}(E), J_\infty) = \mathbb{P}(\mathcal{E}_\infty) = \mathbb{P}(Gr(\mathcal{E}_0)),$$

appearing in the limit, which is precisely an instance of the manifolds considered in [B].

The simplest example of a bundle satisfying our hypotheses is a rank two bundle given as a non-split extension

$$0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$$

of two line bundles with $\deg \mathcal{L}_1 = 1$ and $\deg \mathcal{L}_2 = 0$, and $g(\Sigma) = 3$. The bundle $\mathcal{E}$ can be shown to be simple (see Example 3.7 for details). Morover, the Harder-Narasimhan of this bundle is precisely the inclusion $\mathcal{L}_1 \hookrightarrow \mathcal{E}$, and the associated graded object is

$$Gr(\mathcal{E}) = \mathcal{L}_1 \oplus \mathcal{L}_2.$$

The summands are stable (since they are line bundles), and by assumption have different slopes. Note there are no previous results in the literature proving long-time existence of Calabi flow even in this very simple case.

Once the approximate solution has been found, then the idea is to use the implicit function theorem to find an exact solution, as described above, just as in the elliptic versions of the problem. Here again, we encounter difficulties. First of all, one needs to construct a map between two different Banach spaces, the zeros of which will give a solution to Calabi flow. The natural impulse is to try to define such a map using the operators F_t given by the right hand side of Equation 1.9 (except using the metrics $\omega_{k,l,}(t)$), between two parabolic Sobolev spaces, imitating the elliptic problem. Here, we think of this as defining a map on functions by adding potentials to the metric inside all the operators involved. One needs to reprove the existence of such a map in the time dependent setting, since the result does not follow immediately from the static case, where the spaces are ordinary Sobolev spaces.

A further issue is that the operators F_t do not converge to zero, but rather to an operator F_∞, which by construction is the extremal metric operator (see equation 2.6) employed by Brönnle in [B], which we think of as a map between two ordinary Sobolev spaces. Therefore, in order to obtain a well-defined map at all, it must actually be given as the difference

$$D_t = F_t - F_\infty.$$

Note that the F_t satisfy the property that their pullback by the diffeomorphisms $\widetilde{g}_t$ is precisely the left hand side of Calabi flow. Notice also that D_t, rather than being a map on a single Banach space is now a map on a product of two spaces. Then the inverse function theorem will provide a perturbation giving a solution to the equation

$$F_t\left(\phi_t\right) = F_\infty\left(\phi_\infty\right).$$

Since the equation $F_t\left(\phi_t\right) = 0$ is equivalent to a solution to Calabi flow, we should simultaneously solve $F_\infty\left(\phi_\infty\right) = 0$. Therefore our next guess at a map that will acheive the desired result, is one of the form

$$C_t : W_1 \times V_1 \to W_2 \times V_2$$

where W_1 and W_2 are the parabolic spaces, and V_1 and V_2 ordinary Sobolev spaces, and

$$C_t = (D_t, F_\infty).$$

We may follow the argument of [B] to produce a solution to the second equation, and the hope would be to employ a time-dependent version of the perturbation given there to solve the first. A subtlety of this strategy, is that finding an extremal metric is tantamount to finding a a pair (ϕ_∞, V_∞) where ϕ_∞ is a smooth function (initially in some Sobolev space) and V_∞ is a Hamiltonian Killing vector field, such that the scalar curvature of the metric obtained by adding the potential ϕ_∞ is a Hamiltonian function for V_∞ (see Equation 2.6 below). In other words, one has the additional freedom of perturbing V_∞. This is what is done in [B]. In order to make the inverse function theorem argument work, one needs surjectivity of the linearisation of the map F_∞, which is essentially the Lichnerowicz operator. This will not be true for F_∞ itself, which involves the Hamiltonian function for a certain Hamiltonian Killing vector field arising naturally in [B], but will if we replace this function with a certain perturbation $\widetilde{F}_\infty$ obtained as the Hamiltonian function of a close-by vector field $\widetilde{V}_\infty$. The point here is the the kernel of the operator in question is exactly the space of Hamiltonian Killing fields. By the simplicity assumption on our bundle $\mathcal{E}_0$, there are no non-trivial holomorphic vector fields on $\mathbb{P}(\mathcal{E}_0)$, and therefore there are also none on $\mathbb{P}(\mathcal{E}_t)$ for any t (the Yang-Mills flow preserves the complex gauge orbit). However, the limit $\mathbb{P}(\mathcal{E}_\infty)$ does indeed possess such vector fields, and it is our assumption on the Harder-Narasimhan filtration of $\mathcal{E}_0$ that allows us to characterise these precisely enough to eliminate the kernel.

The problem with this from the point of view of the time dependent part of the map C_t, is that now the operator $F_t - \widetilde{F}_\infty$ is not well defined as a map into the parabolic Sobolev space (W_2 in the schematic above), because it is F_∞ and not $\widetilde{F}_\infty$ to which F_t converges. We therefore have to find a perturbation $\widetilde{F}_t$ for which this holds. However the naive choice (obtained by modifying F_t in a similar way by perturbing the vector field slightly) will result in an operator a zero of which does not produce a solution to Calabi flow on any time interval. This is because it is solutions of the equation

$$\frac{\partial \omega_t}{\partial t} + i\overline{\partial}_{J_t}\partial_{J_t} Scal(\omega_t) + \mathcal{L}_{V_t}\left(\omega_t\right) = 0$$

which pull back to solutions of Calabi flow under $\widetilde{g}_t$, and modifying the vector field V_t destroys this effect.

Our solution is to introduce a cut-off function, so that we obtain a path of operators converging to $\widetilde{F}_\infty$, but (the pullback of which) gives a solution to Calabi flow up to some large time S. We therefore obtain an entire one-parameter family of operators F_t^S as above, and so if we can make the correct perturbation for every S, we will obtain a longtime solution to Calabi flow.

To summarise, our main theorem is the following.

Theorem 1.3. *Let $\mathcal{E} \to \Sigma$ be a simple holomorphic vector bundle over a compact Riemann surface, and assume furthermore that the associated graded object of the Harder-Narasimhan-Filtration only contains stable factors. Let ω_Σ be a constant scalar curvature metric on Σ, h an hermitian metric on the underlying smooth vector bundle E, and J the holomorphic structure on the projectivisation $\mathbb{P}(\mathcal{E})$ induced by $\mathcal{E}$. For $k >> 0$ let $\omega_k(h, J)$ be the Kähler metric on $\mathbb{P}(\mathcal{E})$ defined by $iF_{(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),h)} + k\omega_\Sigma$ in the adiabatic Kähler class $2\pi c_1(\mathcal{V}) + k[\omega_\Sigma]$, where $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the hyperplane bundle, $(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1), h)$ is the Chern connection induced by h, and $\mathcal{V}$ is the vertical tangent bundle of $\mathbb{P}(\mathcal{E})$. Then the Calabi flow on $\mathbb{P}(\mathcal{E})$ starting at the metric $\omega_k(h, J)$ exists for all time.*

1.5. **Outline of the paper.** In Section 2 we give some background on cscK and extremal metrics as well as Calabi flow. We define the Calabi map between certain parabolic Sobolev spaces, to be used in Sections 5 and 6 below, and show that it is well-defined and smooth. In Section 3 we discuss Yang-Mills theory on a smooth, hermitian bundle over a Riemann surface, and in particular the HYM and Yang-Mills flows, and give the critical convergence theorems of Råde and Daskalapoulos. In the process we discuss the Harder-Narasimhan filtration, and describe the geometric conditions we will put on our initial holomorphic bundle $\mathcal{E}_0$. We give some simple examples of bundles over a Riemann surface such that satisfy the condition of our main theorem. In Section 4 we consider the manifolds of interest in this paper, namely the projectivisation $\mathbb{P}(\mathcal{E}_0)$ and the manifolds $\mathbb{P}(\mathcal{E}_t)$ determined by Yang-Mills flow, as well as the limit of these manifolds at infinty $\mathbb{P}(\mathcal{E}_\infty) = \mathbb{P}(Gr(\mathcal{E}_0))$. We construct the Kähler metrics $\omega_k(h_t, J)$, $\omega_k(h, J_t)$, and $\omega_k(h, J_\infty)$ on these manifolds alluded to above, and collect various facts regarding the geometry of this setting, to be used in the sequel.

The heart of the proof is contained in Sections 5 and 6. In Section 5 we construct the approximate metrics $\omega_{k,l}(t)$ (see Theorem 5.1 below). We begin by showing that the metrics $\omega_k(h_t, J)$, $\omega_k(h, J_t)$ actually give formal approximations to Calabi flow up to order 2. We then verify that, for $\omega_k(h, J_t)$ this approximation persists in the parabolic Sobolev norm. The meat of Section 5 is to pass from $\omega_k(h, J_t)$ to the metrics $\omega_{k,2}(t)$ (and their pullbacks $\widehat{\omega}_{k,2}(t)$ under $\widetilde{g}_t$) by adding certain Kähler potentials to the metric. Here we must pass back and forth between moving metric and moving holomorphic structure pictures, as certain calculations are more easily performed in one or the other framework. We construct the various linear parabolic equations that must be solved, and find solutions with parabolic estimates with the aid of Proposition 7.10, to obtain a formal solution up to order k^{-3}. Finally we apply Proposition as well as the linear parabolic estimates, to prove that the this estimate can again be validated in the norm of the parabolic space. In the last subsection of section 5, we show how to perform the inductive argument to obtain this estimate for all orders.

In Section 6, we make the above schematic for our map between two (products of) Banach spaces rigourous, applying Proposition 2.7, gradually building up the correct map following the discussion above. We then consider the linearisation of this map, and prove its surjectivity. We prove a certain estimate on the operator norm of its inverse in the parabolic Sobolev space. Here we are helped by results in the elliptic case from cite [F], and [B]. We finally estimate the non-linear part of our map, giving us all the tools to apply the implicit function theorem, and therefore a longtime solution in the parabolic space, carrying out the sketch given previously. Since we may take the regularity to be as high as we like, we actually obtain a C^∞ solution to the flow for all time.

In the Section 7, the appendix, we give our version of the linear parabolic theory, proving the main existence, regularity and convergence results that we need.

1.6. **Outlook.** Clearly the restrictions we have place on both the bundle $\mathcal{E}$ and on the dimension of the base Σ, are unsatisfactory. We suspect that the hypotheses on $\mathcal{E}$ are an artifact of the proof, and with more work, it should be possible to remove these. It should also be within reach to prove the convergence of the Chen-Sun flow 1.4 to the extremal holomorphic structure in the case considered in the present article. Moreover, given what we have done here, it should be a straightforward problem to prove long-time existence and convergence to the cscK metric when the base manifold (X, ω_X) and the bundle $\mathcal{E}_0$ satisfy the conditions of Hong's theorem in [H], even when X has arbitrary dimension. This is because when $\mathcal{E}_0$ is stable the HYM flow actually converges exponentially, so we avoid most of the difficult technical problems addressed above.

A more difficult problem is to study the case of higher dimensional base manifold when $\mathcal{E}$ is not stable, even if the base has a constant scalar curvature metric. There are two major issues here. First of all, there appears to be no analogue of the result of Råde about the rate of convergence of the Yang-Mills flow, even when the flow converges smoothly. Even more seriously, although the higher dimensonal version of the result of [D] has already been proven (see [DW1], [S]), the Yang-Mills flow can develop singularities along holomorphic subvarieties in infinite time (see [HT]). On the other hand, by [SW] (for surfaces see also [DW2]), this singular set is precisely the singular set of associated graded sheaf $Gr(\mathcal{E}_0)$ of the Harder-Narasimhan-Seshadri double filtration. Note that in higher dimensions this filtration is given by subsheaves rather subbundles, and therefore its graded object is singular in general.

With all of this is mind we can give a slightly more precise version of Donaldson and Chen's conjectures 1.2 and 1.1 for ruled manifolds.

Conjecture 1.4. *Let $(E, h) \to (X, \omega_X)$ be an Hermitan vector bundle over a Kähler manifold with constant scalar curvature, and let A_0 be an integrable, unitary connection determining a holomorphic bundle $\mathcal{E}_0$. Consider the adiabatic metrics $\omega_k(h, J) = F_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)} + ik\pi^*\omega_X$ on $\mathbb{P}(\mathcal{E}_0)$. Then for sufficiently large k the Calabi flow starting from these metrics exists for all time and one of four things occurs:*

- *A cscK metric on $\mathbb{P}(\mathcal{E}_0)$ exists in $[\omega_k(h, J)]$ and Calabi flow converges to the cscK metric.*
- *An extremal metric exists on the manifold $\mathbb{P}(\mathcal{E}_0)$ in $[\omega_k(h, J)]$ and the flow 1.4 converges to the extremal holomorphic structure, which is isomorphic to $\mathbb{P}(\mathcal{E}_0)$. This happens exactly when $\mathcal{E}_0$ splits as a direct sum of stable bundles.*
- *No extremal metric on $\mathbb{P}(\mathcal{E}_0)$ in $[\omega_k(h, J)]$ exists, but an extremal metric exists on $\mathbb{P}(Gr(\mathcal{E}_0))$, which is a complex manifold, the Yang-Mills flow on E converges smoothly without bubbling, and the Chen Sun flow 1.4 converges to the extremal holomorphic structure, which is precisely that of the $\mathbb{P}(Gr(\mathcal{E}_0))$. This happens exactly when the Harder-Narasimhan-Seshadri double filtration $\mathcal{E}_0$ consists of smooth subbundles.*
- *No smooth extremal metric exists even on $\mathbb{P}(Gr(\mathcal{E}_0))$, which is singular, but some sort of singular extremal metric does exist on this space. The Yang-Mills flow converges with singularities along the holomorphic subvariety of X, determined by the sheaf $Gr(\mathcal{E}_0)$. The Chen-Sun flow 1.4 converges smoothly outside of the singular set Sing$(\mathbb{P}(Gr(\mathcal{E}_0))$ to the singular holomorphic structure determined by this space, and this structure determines the singular extremal metric.*

As we have discussed, the first case appears to be relatively straightforward. In the present paper we have made progress towards this problem in the third case when the base is a Riemann surface, and a complete proof in this case ought to be in reach. In higher dimensions, this problem could be approachable using a similar strategy to the one employed in this article, if the requisite analogue of the result of Råde [R] could be obtained. The same is true of the second case listed above, which is an easier version of the same problem. The last case is that of a general unstable bundle in higher dimensions, and appears to require a completely different approach. Here even the extremal metric problem seems not to be properly understood, but to understand what sort of structure should appear in the limit, one might attempt to prove some sort of singular version of the theorem in [B], starting from a direct sum of the singular HYM connections constructed in [BS].

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2. Canonical metrics, the Calabi flow, and operators on parabolic spaces

Our discussion in this subsection is entirely general. Throughout, we fix a Kähler manifold (X, ω, g). We will write J for the almost complex structure associated to the complex structure on X.

2.1. Extremal and cscK metrics.

We define the space of Kähler potentials in the Kähler class $[\omega]$

$$\mathcal{H}_\omega = \{\phi \in C^\infty(\Sigma) | \omega + i\overline{\partial}_J \partial_J \phi > 0\},$$

and will write $\omega_\phi = \omega + i\overline{\partial}_J\partial_J\phi$ for $\phi \in \mathcal{H}_\omega$.

The Ricci curvature of an hermitian metric g on X , is defined to be

$$\rho = tr\left(F(g)\right),$$

where $F(g)$ is the Riemannian curvature of the metric g. Recall that g may be thought of as an element of $\Gamma(T^*X \otimes \overline{T^*X})$. A basic lemma in Kähler geometry is that he Ricci form ρ is equal to $iF_{K_X^*}(g)$, i times the curvature of the induced metric on the anti-canonical bundle K_X^*, which is an element of $\Gamma(K_X \otimes \overline{K}_X)$, and that the latter is fact given by $\frac{\omega^n}{n!} \in \Gamma(K_X \otimes \overline{K}_X)$, so that in local coordinates one may write:

$$\rho(\omega) = i\overline{\partial}_J\partial_J \log \frac{\omega^n}{n!}, \tag{2.1}$$

so that ρ is in fact a closed, real $(1,1)$ form.

The scalar curvature is by definition

$$Scal\left(\omega\right) = \Lambda_\omega \rho\left(\omega\right). \tag{2.2}$$

Here Λ_ω is contraction with the Kähler metric. In local coordinates, for a $(1,1)$-form $\alpha = \alpha_{i,\overline{j}} dz^i \wedge d\overline{z}^j$,

$$\Lambda_\omega \alpha = g^{i\overline{j}} \alpha_{i,\overline{j}}.$$

A metric is called constant scalar curvature Kähler or **cscK**, if the scalar curvature is a constant function. We will write

$$\begin{array}{rcl} Scal_\omega & : & \mathcal{H}_\omega \to C^\infty(X) \\ \phi & \mapsto & Scal\left(\omega + i\overline{\partial}_J\partial_J\phi\right), \end{array}$$

$\omega_\phi \in [\omega]$ is cscK if and only if

$$Scal_\omega(\phi) = c,$$

for some constant c.

The most natural functional on $\mathcal{H}_\omega$ to consider is the **Calabi functional**

$$\begin{aligned} C &: \mathcal{H}_\omega \to \mathbb{R} \\ \phi &\mapsto \int_X (Scal(\omega_\phi))^2 \, dvol_\omega. \end{aligned}$$

A metric is called **extremal** if it is a critical point of this functional. Note that the average scalar curvature is a topological constant:

$$\overline{Scal(\omega)} = \frac{1}{vol(\omega)} \int_X Scal(\omega) \, dvol_\omega = \frac{2\pi n c_1(X) \cup [\omega]^{n-1}}{[\omega]^n}.$$

Then since

$$\int_X (Scal(\omega))^2 \, dvol_\omega = \int_X \left(Scal(\omega) - \overline{Scal(\omega)}\right)^2 dvol_\omega + \int_X \left(\overline{Scal(\omega)}\right)^2 dvol_\omega,$$

cscK metrics are also the minimizers of the functional given by the first term on the right hand .

The Euler-Lagrange equations of C are given by the equation

$$\mathfrak{D}_{(\omega,J)} Scal(\omega) = 0, \tag{2.3}$$

where

$$\mathfrak{D}_{(\omega,J)} : C^\infty(X, \mathbb{C}) \to \Gamma(\Lambda^{0,1}X \otimes T^{1,0}X)$$

is the Lichnerowicz operator defined by

$$\mathfrak{D}_{(\omega,J)} = \overline{\partial}_J(\nabla_g^{1,0}\phi),$$

where $\nabla_g^{1,0}\phi = \frac{1}{2}(\nabla_g\phi - iJ\nabla_g\phi) \in T^{1,0}X$, or in local coordinates

$$\nabla_g^{1,0}\phi = g^{i\bar{j}}\partial_{\bar{j}}\phi,$$

is the $(1,0)$ part of the of $\nabla_g\phi$. In other words, a metric is extremal if and only if the $(1,0)$ part of the gradient is a holomorphic vector field.

We define the Lie algebra

$$\mathfrak{h} = \left\{ V \in H^0\left(T^{1,0}X\right) | V = \nabla_g^{1,0}\phi \text{ for some } \phi \in C^\infty(X, \mathbb{C}) \right\}.$$

We define the space of holomorphy potentials

$$\mathfrak{H} = \ker \mathfrak{D}_{(\omega,J)}.$$

Note ϕ is determined by $\nabla_g^{1,0}\phi$ up to a constant, so $\dim \mathfrak{H} = \dim \mathfrak{h} + 1$. The operator $\mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}$ is self adjoint, so

$$\ker \mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)} = \mathfrak{H}. \tag{2.4}$$

The following is a standard result (see for example [LS]).

Lemma 2.1. *The following two statements hold.*

(i) *A vector field $W \in \Gamma\left(T^{1,0}X\right)$ is holomorphic with zeros if and only if $W \in \mathfrak{h}$, that is if and only if there exists $\phi \in C^\infty(X, \mathbb{C})$ such that $\mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi = 0$, and*

$$\overline{\partial}_J\phi = -\frac{1}{2}\omega(W, -).$$

(ii) *A vector field $V \in \Gamma(TX)$ is a Killing field ($\mathcal{L}_V(g) = 0$) with zeros if and only if there exists a function $\phi \in C^\infty(X, \mathbb{R})$ such that $\mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi = 0$ and*
$$d\phi = -\omega(V, -).$$
This implies more precisely that $V = J\nabla_g\phi$.

In particular, if ReW is a killing field we may choose the function ϕ in (i) above to be real, and the vector field $V - iJV \in \Gamma(T^{1,0}X)$,where V is as in (ii) above, is holomorphic.

Notice that the previous lemma implies that a vector field is a real Killing field with zeros if and only if it is real holomorphic and a Hamiltonian vector field. We therefore make the following definition.

Definition 2.2. A vector field $V \in \Gamma(TX)$ as in (ii) of the preceeding lemma will be called a **Hamiltonian Killing field**. We will write $\mathfrak{ham}(J, g, \omega)$ for the space of such vector fields.

Remark 2.3. If we write $\phi = \phi_1 + i\phi_2$ for $\phi \in C^\infty(X, \mathbb{C})$, then under the isomorphism of real vector bundles $T^{1,0}X \cong TX$,
$$\nabla^{1,0}_g\phi \mapsto \frac{1}{2}(\nabla_g\phi_1 + J\nabla_g\phi_2).$$
Therefore if $W \in \mathfrak{h}$ and $W = \nabla^{1,0}_g\phi$ for ϕ imaginary, then $ReW \in \mathfrak{ham}(J, g, \omega)$, and conversely, so that $\mathfrak{ham}(J, g, \omega)$ is precisely the image under the above isomorphism of elements of $\mathfrak{h}$ with imaginary holonomy potential. We will denote this set by $\mathfrak{k} \subset \mathfrak{h}$.

We may restate the extremal metric condition 2.3 on the Kähler metric ω as $Scal(\omega) \in \mathfrak{H}$. If ω is not extremal, we may try to find a Kähler potential ϕ such that $\omega + i\overline{\partial}_J\partial_J\phi$ is extremal, or in other words such that
$$Scal_\omega(\phi) = Scal(\omega + i\overline{\partial}_J\partial_J\phi) \in \mathfrak{H}.$$
By the previous lemma, in order for the Kähler class $[\omega]$ to admit extremal Kähler metrics, there must be some non-trivial Hamiltonian Killing field $V \in \mathfrak{ham}(J, g, \omega)$ on X, and we must have $V = J\nabla_{g(\omega)}(H_V(\omega))$, so that $H_V(\omega)$ is a Hamiltonian function for V with respect to ω. We may define a function
$$\begin{aligned} H_{\omega,V} \quad &: \quad \mathcal{H}_\omega \to C^\infty(X) \\ \phi \quad &\mapsto \quad H_V(\omega + i\overline{\partial}_J\partial_J\phi), \end{aligned}$$
where $H_V(\omega + i\overline{\partial}_J\partial_J\phi)$ is a Hamiltonian function for V with respect to $\omega + i\overline{\partial}_J\partial_J\phi$. The following is Lemma 20 of Brönnle and computes the function $H_V(\omega + i\overline{\partial}_J\partial_J\phi)$.

Lemma 2.4. *If $V \in \mathfrak{ham}(J, g, \omega)$ and if $\phi \in C^\infty(X)$ is V-invariant, that is, $\mathcal{L}_V\phi = 0$, then we have*
$$H_V(\omega + i\overline{\partial}_J\partial_J\phi) = H_V(\omega) - \frac{1}{2}\nabla_{g(\omega)}(H_V(\omega)) \cdot \nabla_{g(\omega)}\phi.$$

If we fix the the vector field V, we may now we may define the extremal operator with respect to V :
$$\begin{aligned} F_V \quad &: \quad \mathcal{H}_\omega \to C^\infty(X) \\ \phi \quad &\mapsto \quad Scal(\omega + i\overline{\partial}_J\partial_J\phi) - H_V(\omega + i\overline{\partial}_J\partial_J\phi). \end{aligned}$$
In other words
$$F_V = Scal_\omega + H_{\omega,V}.$$

By Lemma 2.4, if we write $\mathcal{H}_{\omega,V} \subset \mathcal{H}_\omega$ for the subset of V-invariant Kähler potentials, we may write

$$
\begin{aligned}
F_V \quad &: \quad \mathcal{H}_{\omega,V} \to C^\infty(X) \\
\phi \quad &\mapsto \quad Scal(\omega + i\overline{\partial}_J\partial_J\phi) - H_V(\omega) + \frac{1}{2}\nabla_{g(\omega)}H_V(\omega)\cdot\nabla_{g(\omega)}\phi \\
&= \quad Scal(\omega + i\overline{\partial}_J\partial_J\phi) - H_V(\omega) + \frac{1}{2}\mathcal{L}_{\nabla_{g(\omega)}H_V(\omega)}(\phi) \qquad (2.5)
\end{aligned}
$$

Clearly then the extremal metric condition can be restated as

$$F_V(\phi) = 0, \qquad (2.6)$$

since this says precisely that $Scal(\omega + i\overline{\partial}_J\partial_J\phi)$ is a Hamiltonian function for V with respect to $\omega + i\overline{\partial}_J\partial_J\phi$, and therefore lies in $\mathfrak{H}$. Since the choice of V was arbitrary, we may more generally consider the map

$$
\begin{aligned}
F \quad &: \quad \mathfrak{ham}(J,g,\omega) \times \mathcal{H}_\omega \to C^\infty(X) \\
(V,\phi) \quad &\mapsto \quad Scal(\omega + i\overline{\partial}_J\partial_J\phi) - H_V(\omega + i\overline{\partial}_J\partial_J\phi).
\end{aligned}
$$

2.2. **Linearisations.** We will now give the linearisations of the scalar curvature and extremal metric operators. Recall that the Lichnerowicz operator satisfies the following formula:

$$\mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi = \Delta^2_\omega(\phi) + g_\omega(\nabla Scal(\omega), \nabla\phi) + n(n-1)\frac{i\overline{\partial}_J\partial_J(\phi)\wedge\rho_\omega\wedge\omega^{n-2}}{\omega^n}. \qquad (2.7)$$

We will write $d_0(Scal)_\omega$ for the derivative at 0 (the linearisation) of the map $Scal_\omega$. A formula for the linearisation of the scalar curvature is given by the following lemma, which is Lemma 2.1 of [F].

Lemma 2.5. *Let (X,ω) be a Kähler manifold of dimension n. Let $V \subset L^2_{d+4}$ be the L^2_{d+4} completion of an open set $\mathcal{H}_\omega \subset C^\infty(X)$. The map*

$$Scal_\omega : V \to L^2_d$$

defined by $\phi \to Scal(\omega_\phi)$ is smooth as a map of Banach spaces when $d > n-2$..

$$
\begin{aligned}
d_0(Scal)_\omega &= \mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi + g_\omega(\nabla Scal(\omega), \nabla\phi) \\
&= \Delta^2_\omega(\phi) - Scal_\omega(0)\Delta_\omega(\phi) + n(n-1)\frac{i\overline{\partial}_J\partial_J(\phi)\wedge\rho_\omega\wedge\omega^{n-2}}{\omega^n},
\end{aligned}
$$

so that in particular if ω has constant scalar curvature, then

$$d_\omega Scal(\phi) = \mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi,$$

and if ω is Kähler-Einstein with Einstein constant λ, then

$$d_0(Scal)_\omega = \Delta^2_\omega(\phi) - \lambda\Delta_\omega(\phi).$$

Lemma 2.6. *If we fix $V \in \mathfrak{ham}(J,g,\omega)$, the linearisation of the map F_V at 0 is given by*

$$
\begin{aligned}
(dF_V)_0 \quad &: \quad \mathcal{H}_{\omega,V} \to C^\infty(X) \\
\phi \quad &\mapsto \quad \mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi - \frac{1}{2}g_\omega(\nabla Scal(\omega), \nabla\phi) + \frac{1}{2}\nabla_{g(\omega)}H_V(\omega)\cdot\nabla_{g(\omega)}\phi \\
&= \quad \mathfrak{D}^*_{(\omega,J)}\mathfrak{D}_{(\omega,J)}\phi - \frac{1}{2}\mathcal{L}_{\nabla Scal(\omega)}(\phi) + \frac{1}{2}\mathcal{L}_{\nabla_{g(\omega)}H_V(\omega)}(\phi).
\end{aligned}
$$

This follows automatically from Lemmas 2.5 and 2.4.

2.3. **Calabi flow.** Let X be a complex manifold with holomorphic structure J. Fix a Kahler form ω which is compatible with J determining a Kähler triple (J, ω, g_ω) on X.

There is a functional

$$M : \mathcal{H}_\omega \to \mathbb{R},$$

called the **Mabuchi energy,** the norm square of whose gradient is

$$\int_X \left(Scal(\omega_\phi) - \overline{Scal(\omega_\phi)}\right)^2 dvol_\omega.$$

In other words, it is the anti-derivative of the closed one-form

$$(dM)_\phi\,(\psi) = \int_X (Scal(\omega_\phi) - \overline{Scal(\omega_\phi)})dvol_\omega.$$

Note, that there is normally a minus sign in the above formula, but we have defined the space $\mathcal{H}_\omega$ using the operator $\overline{\partial}_J\partial_J$ rather than $\partial_J\overline{\partial}_J$, as is customary, so our $\mathcal{H}_\omega$ is minus the usual space of Kähler potentials. The negative gradient flow of this functional is the equation

$$\frac{\partial\phi_t}{\partial t} = -(Scal(\omega_{\phi_t}) - \overline{Scal(\omega_{\phi_t})}), \tag{2.8}$$

and writing $\omega_t = \omega + i\overline{\partial}_J\partial_J\phi_t$ we see that this equation is equivalent to equation 1.6, so we may also refer to it as Calabi flow.

In this paper we will have occasion to consider the action of the diffeomorphism group diff(X), on the set of complex structures $\mathcal{J}_X$ on X given by $\xi \cdot J = d\xi \circ J \circ (d\xi)^{-1}$. In terms of $\bar{\partial}$ operators the action is given by $\bar{\partial}_J \circ \xi^* = \xi^* \circ \bar{\partial}_{\xi \cdot J}$. Since $\xi^* \circ d_X = d_X \circ \xi^*$ we also have $\partial_J \circ \xi^* = \xi^* \circ \partial_{\xi \cdot J}$. It is clear that triple $(\phi^{-1} \cdot J, \phi^*\omega, \phi^* g_\omega)$ is a Kähler triple, and by construction, the map $\phi : (X, J) \to (X, \phi \cdot J)$ is holomorphic.

If ω_t is a path of Kahler forms on the moving complex manifold (X, J_t), then given a one-parameter family of diffeomorphisms ξ_t, $\xi_t^*(\omega(t))$ is a path of Kähler forms on the fixed complex manifold (X, J).

Notice that

$$Ric(\xi_t^*(\omega(t))) \quad = \quad \xi_t^*\left(Ric\omega(t)\right),$$

and so

$$Scal\left(\xi_t^*(\omega(t))\right) = \xi_t^*\Lambda_{\omega(t)}\left(Ric\omega(t)\right) = \xi_t^* Scal\left(\omega(t)\right),$$

and we therefore obtain

$$i\bar{\partial}_J\partial_J\left(Scal\left(\xi_t^*(\omega(t))\right)\right) = \xi_t^*\left(i\partial_{J_t}\partial_{J_t}Scal\left(\omega(t)\right)\right)$$

Secondly a standard fact is that

$$\frac{\partial\xi_t^*(\omega(t))}{\partial t} = \xi_t^*\left(\frac{\partial\omega(t)}{\partial t} + \mathcal{L}_{V_t}\omega(t)\right),$$

where V_t is the (time-dependent) flow of the diffeomorphisms ξ_t.

Throughout the paper we will consider the equation:

$$\frac{\partial\omega(t)}{\partial t} + i\bar{\partial}_{J_t}\partial_{J_t}Scal\left(\omega(t)\right) + \mathcal{L}_{V_t}\omega(t) = 0 \tag{2.9}$$

on the moving complex manifold (X, J_t). We will call this equation **Calabi flow up to diffeomorphisms**, because using the above facts one sees that a solution to this equation is a equivalent to the fact that $\xi_t^*(\omega(t))$ solves Calabi flow 1.6 on (X, J).

2.4. **The Calabi operator and its linearisation.** Recall the parabolic Sobolev spaces $W^0_{4,p+1,w_\varepsilon(t)}$ and $W_{4,p,q,w_\varepsilon(t)}$ from Section 7. The parabolic analogue of Lemma 2.5 is the following.

Lemma 2.7. *Let X be a compact manifold and (J_t, g_t, ω_t) a family of Kähler structures on X, converging smoothly to a Kähler structure $(J_\infty, g_\infty, \omega_\infty)$ on X, such that*

$$\|J_t - J_\infty\|_{W_{4,p+1,q,w_\varepsilon(t)}(g_\infty)}, \|\omega_t - \omega_\infty\|_{W_{4,p+1,q,w_\varepsilon(t)}(g_\infty)} < \infty,$$

and

$$\|Scal(\omega_t) - Scal(\omega_\infty)\|_{W_{4,p,q-1,w_\varepsilon(t)}(g_\infty)} < \infty.$$

Then writing $Scal_{\omega_t}(\phi_t + \phi_\infty) = Scal(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty))$ and $Scal_{\omega_\infty}(\phi_\infty) = Scal(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty)$, there is a well-defined, differentiable map

$$\begin{aligned} \frac{\partial}{\partial t} + Scal_{\omega_t} - Scal_{\omega_\infty} \quad &: \quad W^0_{4,p+1,q,w_\varepsilon(s)}(g_\infty) \times L^2_{4(p+1)}(g_\infty) \to W_{4,p,q-1,w_\varepsilon(s)}(g_\infty) \\ (\phi_t, \phi_\infty) \quad &\mapsto \quad \frac{\partial \phi_t}{\partial t} + Scal_{\omega_t}(\phi_t + \phi_\infty) - Scal_{\omega_\infty}(\phi_\infty), \end{aligned}$$

whenever $q < p - (n-2)/4$. Moreover, the derivative of this map at 0 is given by

$$\begin{aligned} (\phi_t, \phi_\infty) \quad \mapsto \quad & \frac{\partial \phi_t}{\partial t} + \mathfrak{D}^*_{\omega_t}\mathfrak{D}_{\omega_t}(\phi_t + \phi_\infty) - \mathfrak{D}^*_{\omega_\infty}\mathfrak{D}_{\omega_\infty}(\phi_\infty) \\ & -\frac{1}{2}\left(g_t\left(\nabla_{g_t} Scal(\omega_t), \nabla_{g_t}(\phi_t + \phi_\infty)\right) - g_\infty\left(\nabla_{g_\infty} Scal(\omega_\infty), \nabla_{g_\infty}\phi_\infty\right)\right). \end{aligned}$$

Proof. First of all, if $\phi_t \in W^0_{4,p+1,q,w_\varepsilon(s)}(g_\infty)$, then clearly

$$\begin{aligned} & \left\|\frac{\partial \phi_t}{\partial t}\right\|_{W_{4,p,q-1,w_\varepsilon(t)}(g_\infty)} \\ = \ & \sum_{j=0}^{q-1} \int_0^\infty |w_\varepsilon(t)|^2 \left\|\frac{\partial^j}{\partial t}\frac{\partial \phi_t}{\partial t}\right\|_{L^2_{4(p-j)}(g_\infty)} = \sum_{i=1}^{q} \int_0^\infty |w_\varepsilon(t)|^2 \left\|\frac{\partial^i}{\partial t}\phi_t\right\|^2_{L^2_{4(p+1-i)}(g_\infty)} \\ \leq \ & \left\|\frac{\partial \phi_t}{\partial t}\right\|_{W_{4,p+1,q,w_\varepsilon(t)}(g_\infty)} < \infty, \end{aligned}$$

so $\partial_t\phi_t \in W_{4,p,q-1,w_\varepsilon(s)}(g_\infty)$.

It remains to show

$$Scal_{\omega_t}(\phi_t + \phi_\infty) - Scal_{\omega_\infty}(\phi_\infty) \in W_{4,p,q,w_\varepsilon(t)}(g_\infty).$$

If $\rho_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}$ and $\rho_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}$ are the Ricci curvatures of $\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)$ and $\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty$ respectively, we may write

$$\begin{aligned} & \left(\rho_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)} - \rho_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\right) \wedge \left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)^{n-1} \\ & +\rho_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty} \wedge \left(\left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)^{n-1} - \left(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right) \\ = \ & \rho_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)} \wedge \left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)^{n-1} - \rho_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty} \wedge \left(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1} \\ = \ & Scal_{\omega_t}(\phi_t + \phi_\infty) \wedge \left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)^{n} - Scal_{\omega_t}(\phi_\infty)\left(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n} \\ = \ & \left(Scal_{\omega_t}(\phi_t + \phi_\infty) - Scal_{\omega_t}(\phi_\infty)\right)\left(\left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)\right)^{n} \\ & +Scal_{\omega_t}(\phi_\infty)\left(\left(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t + \phi_\infty)\right)^{n} - \left(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n}\right), \end{aligned}$$

so that for each j we have

$$\begin{aligned}
&\partial_t^j\left((Scal_{\omega_t}(\phi_t+\phi_\infty)-Scal_{\omega_t}(\phi_\infty))\right)\\
= \;&\partial_t^j\left(\frac{\left(\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}-\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\right)\wedge\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}}{\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)\right)^n}\right)\\
&+\partial_t^j\left(\frac{\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\wedge\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right)}{\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)\right)^n}\right)\\
&-\partial_t^j\left(\frac{Scal_{\omega_t}(\phi_\infty)\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n}-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n}\right)}{\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)\right)^n}\right).
\end{aligned}$$

Notice that this calculation makes sense, because by construction for each j in the stated range

$$\partial_t^j\left(i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right),$$

is continuous so the products in the above formulae involving this quantity are meaningful. We therefore have:

$$\begin{aligned}
&\|Scal_{\omega_t}(\phi_t+\phi_\infty)-Scal_{\omega_\infty}(\phi_\infty)\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_\infty)}\\
= \;&\sum_{j=0}^{q-1}\int_0^\infty|w_\varepsilon(t)|^2\left\|\frac{\partial^j}{\partial t}Scal_{\omega_t}(\phi_t+\phi_\infty)-Scal_{\omega_\infty}(\phi_\infty)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
\le \;&C\sum_{j=0}^{q-1}\int_0^\infty|w_\varepsilon(t)|^2\left\|\partial_t^j\left(\frac{\left(\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}-\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\right)\wedge\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}}{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
&+C\sum_{j=0}^{q-1}\int_0^\infty|w_\varepsilon(t)|^2\left\|\partial_t^j\left(\frac{\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\wedge\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right)}{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
&+C\sum_{j=0}^{q-1}\int_0^\infty|w_\varepsilon(t)|^2\left\|\partial_t^j\left(\frac{Scal_{\omega_\infty}(\phi_\infty)\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n}-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n}\right)}{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
\le \;&C_1+C_2\sum_{j=0}^{q-1}\int_0^\infty|w_\varepsilon(t)|^2\left\|\partial_t^j\left(\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}-\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\right)\right\|_{L^2_{4(p-j)}(g_\infty)},
\end{aligned}$$

where we have used the assumption

$$\|\omega_t-\omega_\infty\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\infty)},\|\phi_t\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\infty)},\|J_t-J_\infty\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\infty)}<\infty.$$

More specifically, since if we write

$$\partial_{J_t}=\partial_{J_\infty}+a_t^{1,0},\overline{\partial}_{J_t}=\overline{\partial}_{J_\infty}+a_t^{0,1}$$

where $a_t^{1,0}\in\Omega^{1,0}(End\,(\underline{\mathbb{C}}))$, $a_t^{0,1}\in\Omega^{0,1}(End\,(\underline{\mathbb{C}}))$, with $a_t^{1,0},a_t^{0,1}\in W_{4,p+1,q,w_\varepsilon(s)}(g_\infty)$, then

$$\begin{aligned}
&\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^n\right)\\
= \;&((\omega_t-\omega_\infty))\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}+\cdots+\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right)
\end{aligned}$$

$$+\left(ia_t^{0,1}\circ\partial_{J_\infty}(\phi_t+\phi_\infty)\right)\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}+\cdots+\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right)$$
$$+\left(\left(\overline{\partial}_{J_\infty}\circ a_t^{1,0}+ia_t^{1,0}\wedge a_t^{0,1}\right)(\phi_t+\phi_\infty)\right)\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}+\cdots+\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right)$$
$$+\left(i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_t\right)\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}+\cdots+\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right).$$

This says in particular that for each j,

$$\left\|\partial_t^j\left(\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)\right)^n\right)\right\|_{L^2_{4(p-j)}(g_\infty)}$$

is uniformly bounded in time. Note also that dividing by the Kähler form is the same thing is taking the inner product, so we may also apply the Sobolev multiplication theorem

$$\|T_1\cdot T_2\|_{L^2_{4(p-j)}(g_\infty)}\le\|T_1\|_{L^2_{4(p-j)}(g_\infty)}\|T_2\|_{L^2_{4(p-j)}(g_\infty)},$$

for two tensors T_1 and T_2, and $\cdot$ is any algebraic operation defined using tensor product and contraction. This has also been used in the estimate above. The same calculation applies to

$$\left(\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)^{n-1}\right),$$

and all other quantities involved in the integrals above involving these differences are bounded in the appropriate norms, these two integrals are finite. It remains to prove finiteness of the final integral. We may write

$$\begin{aligned}\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}&=\rho_{\omega_t}+i\overline{\partial}_{J_t}\partial_{J_t}\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)\\&=\rho_{\omega_t}+i\overline{\partial}_{J_t}\partial_{J_t}\log\left(1+\sum_{i=1}^n\frac{(\omega_t)^{n-i}\wedge\left(i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^i}{\omega_t^n}\right),\end{aligned}$$

and similarly

$$\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}=\rho_{\omega_\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\log\left(1+\sum_{i=1}^n\frac{(\omega_\infty)^{n-i}\wedge\left(i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^i}{\omega_\infty^n}\right)$$

and so obtain:

$$\begin{aligned}&\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}-\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\\=\;&\rho_{\omega_t}-\rho_{\omega_\infty}\\&+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)-\log\left(\frac{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}{\omega_\infty^n}\right)\right)\\&+ia_t^{0,1}\circ\partial_{J_\infty}\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)\right)+\overline{\partial}_{J_\infty}\circ a_t^{1,0}\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)\right)\\&+ia_t^{1,0}\wedge a_t^{0,1}\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)\right).\end{aligned}$$

We get an estimate

$$\begin{aligned}
&\sum_{j=0}^{q-1}\int_0^\infty |w_\varepsilon(t)|^2 \left\|\partial_t^j\left(\left(\rho_{\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)}-\rho_{\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty}\right)\wedge\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^{n-1}\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
\leq\ & C+\sum_{j=0}^{q-1}\int_0^\infty |w_\varepsilon(t)|^2\left\|\partial_t^j\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\omega_t^n}\right)-\log\left(\frac{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}{\omega_\infty^n}\right)\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
=\ & C+\sum_{j=0}^{q-1}\int_0^\infty |w_\varepsilon(t)|^2\left\|\partial_t^j\left(\log\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}\right)+\log\left(\frac{\omega_\infty^n}{\omega_t^n}\right)\right)\right\|_{L^2_{4(p-j)}(g_\infty)}\\
\leq\ & C+\sum_{j=0}^{q-1}\int_T^\infty |w_\varepsilon(t)|^2\left\|\partial_t^j\left(\Gamma\left(\omega_t,\omega_\infty,\phi_t,\phi_\infty\right)+\Pi\left(\omega_t,\omega_\infty\right)\right)\right\|_{L^2_{4(p-j)}(g_\infty)}
\end{aligned}$$

where we have set:

$$\begin{aligned}
\Gamma\left(\omega_t,\omega_\infty,\phi_t,\phi_\infty\right) &= \sum_{i=1}^\infty(-1)^{i+1}\frac{\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}-1\right)^i}{i},\\
\Pi\left(\omega_t,\omega_\infty\right) &= \sum_{i=1}^\infty(-1)^{i+1}\frac{\left(\frac{\omega_\infty^n}{\omega_t^n}-1\right)^i}{i},
\end{aligned}$$

and used the Taylor expansion

$$\ln x=\sum_{i=1}^\infty(-1)^{i+1}\frac{(x-1)^i}{i},\ \ |x|<1,$$

and T is taken sufficiently large so that this expansion is valid, which can be done since

$$\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n}{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n},\frac{\omega_\infty^n}{\omega_t^n}\to 1$$

as $t\to\infty$. Pointwise we may calculate

$$\begin{aligned}
&\left|\left\|\partial_t^j\left(\Gamma\left(\omega_t,\omega_\infty,\phi_t,\phi_\infty\right)+\Pi\left(\omega_t,\omega_\infty\right)\right)\right\|\right|_{L^2_{4(p-j)}(g_\infty)}\\
\leq\ & C\sum_{i=1}^\infty\left\|\partial_t^j\left(\frac{\left(\omega_t+i\overline{\partial}_{J_t}\partial_{J_t}(\phi_t+\phi_\infty)\right)^n-\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}{\left(\omega_\infty+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)\right)^n}\right)^i\right\|_{L^2_{4(p-j)}(g_\infty)}\\
&+C\sum_{i=1}^\infty\left\|\partial_t^j\left(\frac{\omega_\infty^n-\omega_t^n}{\omega_t^n}\right)^i\right\|_{L^2_{4(p-j)}(g_\infty)}\\
\leq\ & C\left(\sum_{i=1}^\infty\left\|\partial_t^j\left(\omega_\infty-\omega_t\right)\right\|_{L^2_{4(p-j)}(g_\infty)}+\left\|\partial_t^j a_t^{(1,0)}\right\|_{L^2_{4(p-j)}(g_\infty)}+\left\|\partial_t^j a_t^{(0,1)}\right\|_{L^2_{4(p-j)}(g_\infty)}\right)\\
&+C\left(+\left\|\partial_t^j\left(a_t^{1,0}\wedge a_t^{0,1}\right)\right\|_{L^2_{4(p-j)}(g_\infty)}+\left\|\partial_t^j\phi_t\right\|_{L^2_{4(p-j)}(g_\infty)}\right),
\end{aligned}$$

so finally we obtain

$$\|Scal_{\omega_t}(\phi_t+\phi_\infty)-Scal_{\omega_\infty}(\phi_\infty)\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_\infty)}$$

$$\leq \quad C_1 + C_2 \sum_{j=0}^{q-1} \int_T^\infty |w_\varepsilon(t)|^2 \left\| \left(\left| \partial_t^j (\omega_\infty - \omega_t) \right| + \left| \partial_t^j a_t^{(1,0)} \right| + \left| \partial_t^j a_t^{(0,1)} \right| + \left| \partial_t^j \left(a_t^{1,0} \wedge a_t^{0,1} \right) \right| + \left| \partial_t^j \phi_t \right| \right) \right\|_{L^2_{4(p-j)}(g_\infty)}$$

proving the first claim, namely that the map is well-defined. To prove differentiability, it suffices to compute all directional derivatives

$$\begin{aligned} & \frac{d}{dw} \frac{\partial (\psi_t + w(\phi_t))}{\partial t} + Scal_{\omega_t}(\psi_t + w(\phi_t + \phi_\infty)) - Scal_{\omega_\infty}(\psi_\infty + w(\phi_\infty))|_{w=0} \\ = \quad & \frac{d}{dw} \frac{\partial (\psi_t + w(\phi_t))}{\partial t} + Scal_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t}(w(\phi_t + \phi_\infty)) - Scal_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty}(w(\phi_\infty))|_{w=0}, \end{aligned}$$

for all pairs of 2-tuples

$$(\psi_t, \psi_\infty), (\phi_t, \phi_\infty) \in W^0_{4,p+1,q,w_\varepsilon(s)}(g_\infty) \times L^2_{4(p+1)}(g_\infty),$$

and prove their continuity. By a calculation formally the same as those of Section 5 below shows that when w is sufficiently small, there is an expansion of the form:

$$\begin{aligned} & Scal_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t}(w(\phi_t + \phi_\infty)) \\ = \quad & Scal(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t) + w \left(\mathfrak{D}^*_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t} \mathfrak{D}_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t} (\phi_t + \phi_\infty) \right) \\ & -\frac{1}{2} w g_{\psi_t} \left(\nabla_{g_{\psi_t}} Scal(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t), \nabla_{g_{\psi_t}} (\phi_t + \phi_\infty) \right) + \mathcal{O}(w^2), \end{aligned}$$

and similarly

$$\begin{aligned} & Scal_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty}(w(\phi_\infty)) \\ = \quad & Scal(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty) + w \left(\mathfrak{D}^*_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty} \mathfrak{D}_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty} (\phi_\infty) \right) \\ & -\frac{1}{2} w g_{\psi_\infty} \left(\nabla_{g_{\psi_\infty}} Scal(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty), \nabla_{g_{\psi_\infty}} \phi_\infty \right) + \mathcal{O}(w^2), \end{aligned}$$

where g_{ψ_t} and g_{ψ_∞} are the Riemannian metrics associated to $\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t$ and $\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty$ respectively. Therefore, the directional derivative of $\frac{\partial}{\partial t} + Scal_{\omega_t} - Scal_{\omega_\infty}$ at (ψ_t, ψ_∞) in the direction of (ϕ_t, ϕ_∞) is given by

$$\begin{aligned} & \partial_{(\phi_t, \phi_\infty)} \left(\frac{\partial}{\partial t} + Scal_{\omega_t} - Scal_{\omega_\infty} \right) (\psi_t, \psi_\infty) \\ & \frac{\partial \phi_t}{\partial t} + \mathfrak{D}^*_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t} \mathfrak{D}_{\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t} (\phi_t + \phi_\infty) - \mathfrak{D}^*_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty} \mathfrak{D}_{\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty} (\phi_\infty) \\ & -\frac{1}{2} g_{\psi_t} \left(\nabla_{g_{\psi_t}} Scal(\omega_t + i\overline{\partial}_{J_t}\partial_{J_t}\psi_t), \nabla_{g_{\psi_t}} (\phi_t + \phi_\infty) \right) \\ & +\frac{1}{2} g_{\psi_\infty} \left(\nabla_{g_{\psi_\infty}} Scal(\omega_\infty + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\psi_\infty), \nabla_{g_{\psi_\infty}} \phi_\infty \right). \end{aligned}$$

This assignment is continuous (in fact uniformly continuous) in (ψ_t, ψ_∞) by Lemma 4.19 below, where we note that although the proof there is give for particular metric on a projective bundle, the proof only uses the stated properties of our path of metrics and holomorphic structures. This proves that the map $\frac{\partial}{\partial t} + Scal_{\omega_t} - Scal_{\omega_\infty}$ is differentiable, and furthermore that the derivative is given by the continuous map

$$d \left(\frac{\partial}{\partial t} + Scal_{\omega_t} - Scal_{\omega_\infty} \right) : W^0_{4,p+1,q,w_\varepsilon(s)}(g_\infty) \times L^2_{4(p+1)}(g_\infty) \to \mathcal{L} \left(W^0_{4,p+1,q,w_\varepsilon(s)}(g_\infty) \times L^2_{4(p+1)}(g_\infty), L^2_{4p}(g_\infty) \right)$$

into the space of linear maps between the range and domain defined by the above formula. In particular

$$
\begin{aligned}
& d_{(0,0)}\left(\frac{\partial}{\partial t}+Scal_{\omega_t}-Scal_{\omega_\infty}\right) \\
= \quad & \frac{\partial \phi_t}{\partial t}+\mathfrak{D}^*_{\omega_t}\mathfrak{D}_{\omega_t}\left(\phi_t+\phi_\infty\right)-\mathfrak{D}^*_{\omega_\infty}\mathfrak{D}_{\omega_\infty}\left(\phi_\infty\right) \\
& -\frac{1}{2}g_t\left(\nabla_{g_t}Scal(\omega_t),\nabla_{g_t}(\phi_t+\phi_\infty)\right)+\frac{1}{2}g_\infty\left(\nabla_{g_\infty}Scal(\omega_\infty),\nabla_{g_\infty}\phi_\infty\right),
\end{aligned}
$$

as required. □

3. Yang Mills connections and the Yang-Mills flow on Riemann Surfaces

3.1. **Notation.** Throughout the rest of the paper we will let Σ be a Riemann surface and (E,h) an hermitian vector bundle. We will use the convention h is linear in the first entry and conjugate linear in the second. We will write $\tilde{\pi}:E\to\Sigma$ for the associated projection map. From here on out we will fix a Kähler metric g_Σ on Σ with associated Kähler form ω_Σ. Later we will require ω_Σ to be a constant scalar curvature metric, but in this section ω_Σ will be arbitrary. A holomorphic structure on E will be thought of as an operator $\bar{\partial}_E:\Omega^0(E)\to\Omega^{0,1}(E)$ such that $\bar{\partial}_E^2=0$. We will denote the space of such operators by $\mathcal{A}_{\mathrm{hol}}(E)$. If ∇_A is an integrable, h-unitary connection on E, then its $(0,1)$ part $\bar{\partial}_A$ is a holomorphic structure on E such that the Chern connection $(\bar{\partial}_A,h)$ is ∇_A. We will denote the space of such connections by $\mathcal{A}_h^{1,1}(E)$ (because they have $(1,1)$ curvature). For any such connection we will denote the corresponding holomorphic bundle $(E,\bar{\partial}_A)$ by $\mathcal{E}$. More generally, we will always denote smooth vector bundles by ordinary letters, and holomorphic vector bundles by script letters.

We will denote by $\mathcal{G}^{\mathbb{C}}$ the group of complex gauge transformations of E, that is, the set of complex linear bundle automorphisms $g:E\to E$. This group acts on $\mathcal{A}_{\mathrm{hol}}(E)$ by

$$g\cdot\bar{\partial}=g\circ\bar{\partial}\circ g^{-1}. \tag{3.1}$$

Note that $\mathcal{G}^{\mathbb{C}}$ also acts on the space of Hermitian metrics on E by

$$g\cdot h(u,v)=h(g^{-1}(u),g^{-1}(v)). \tag{3.2}$$

The group $\mathcal{G}$ of h-unitary gauge transformations is the subgroup of $\mathcal{G}^{\mathbb{C}}$ such that $g\cdot h=h$. $\mathcal{G}$ also acts on the space on $\mathcal{A}_h^{1,1}(E)$ by

$$g\cdot\nabla_A=g\circ\nabla_A\circ g^{-1}. \tag{3.3}$$

Since $\mathcal{A}_{\mathrm{hol}}(E)\simeq\mathcal{A}_h^{1,1}(E)$, this action extends to an action of $\mathcal{G}^{\mathbb{C}}$. Note, however that this latter actions is not by conjugation.

Given two Hermitian metrics, h_1 and h_2, we may define an endomorphism $h_2^{-1}h_1$ by

$$h_1(u,v)=h_2(u,h_2^{-1}h_1(v)). \tag{3.4}$$

On the other hand given an Hermitian metric h and an endomorphism k, we may define a new metric hk by

$$hk(u,v)=h(k(u),v). \tag{3.5}$$

3.2. **Yang mills connections, and split and simple vector bundles.** We define the **Yang-Mills functional**

$$YM : \mathcal{A}_h^{1,1}(E)/\mathcal{G} \to \mathbb{R}$$

by

$$YM(A) = \int_\Sigma |F_A|^2 \, dvol_{\omega_\Sigma}. \tag{3.6}$$

The critical points of this functional, called **Yang-Mills connections** on the bundle E, are solutions to the equation

$$d_A^* F_A = 0, \tag{3.7}$$

. By the Kahler identities this is equivalent to

$$d_A \Lambda_\omega F_A = 0 \tag{3.8}$$

This last equation easily implies that there is a splitting of Hermitian, holomorphic bundles:

$$\begin{aligned}(\mathcal{E}, h) &= (E, \bar\partial_A, h) = (E_1, \bar\partial_{A_1}, h_1) \oplus \cdots \oplus (E_q, \bar\partial_{A_q}, h_q) \\ &= (\mathcal{E}_1, h_1) \oplus \cdots \oplus (\mathcal{E}_q, h_q),\end{aligned}$$

where the Chern connections $\nabla_{A_i} = (\bar\partial_{A_i}, h_i)$ satisfy the equations $\Lambda_\omega F_{A_i} = \mu(\mathcal{E}_i) \, Id_{E_i}$, where

$$\mu(\mathcal{E}) = \frac{\int_\Sigma c_1(\mathcal{E}) \, dvol_{\omega_\Sigma}}{rk\mathcal{E}}$$

is called the **slope**. Clearly in this case the connection also splits as $\nabla_A = \nabla_{A_1} \oplus \cdots \oplus \nabla_{A_q}$. The connections A_i are called **Hermitian-Yang-Mills(HYM)**. The existence of an Hermitian-Yang-Mills connection is equivalent (by the Donaldson-Uhlenbeck-Yau theorem) to the slope poly-stability of the bundle. A bundle is **(poly)stable** if (it is a direct sum sum of bundles of the same slope for which) any proper sub-bundle has smaller slope.

Definition 3.1. *A bundle $\mathcal{E} \to \Sigma$ is called simple*

$$H^0(End(\mathcal{E})) = \mathbb{C} \cdot Id_E \tag{3.9}$$

The following lemma is standard.

Lemma 3.2. *A stable vector bundle is in particular simple.*

Lemma 3.3. *Let $\mathcal{E}$ be a simple holomorphic vector bundle with underlying smooth bundle E. Let h be a an hermitian metric on E, and write $A = (\overline{\partial}_{\mathcal{E}}, h)$ for the Chern connection, and $A^{End(E)}$ the induced connection on $End(E)$.*

$$\mathbb{C} \cdot Id_E = \ker \Delta_{A^{EndE}} = \ker d_{A^{End(E)}} \subset H^0(End(\mathcal{E})).$$

Proof. Since $\Delta_{A^{EndE}} = d^*_{A^{End(E)}} d_{A^{End(E)}}$, clearly $\ker \Delta_{A^{EndE}} = \ker d_{A^{End(E)}}$. On the other hand we have

$$d_{A^{End(E)}} = \partial^{End(E)}_{(\mathcal{E},h)} + \overline{\partial}^{End(\mathcal{E})}_A,$$

so $\ker d_{A^{End(E)}} \subset \ker \overline{\partial}^{End(\mathcal{E})}_A = H^0(End(\mathcal{E})) = \mathbb{C} \cdot Id_E$, by simplicity. Since clearly $\mathbb{C} \cdot Id_E \subset \ker d_{A^{End(E)}}$ also, we obtain the result. $\square$

Lemma 3.4. *Let $\mathcal{E}$ be a holomorphic vector bundle (with underlying smooth bundle E) such that*

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$$

where each $\mathcal{E}_i$ is stable, $\mu(\mathcal{E}_1) > \cdots > \mu(\mathcal{E}_l)$ (so that in particular the slopes of the $\mathcal{E}_i$ are all different). Let h be a an hermitian metric on E, and write $A = (\overline{\partial}_{\mathcal{E}}, h)$ for the Chern connection, and $A^{End(E)}$ the induced connection on $End(E)$. Then

$$\mathbb{C}^l = \ker \Delta_{A^{EndE}} = \ker d_{A^{End(E)}} \subset H^0(End(\mathcal{E})).$$

In other words, the covariantly constant sections (and so also the elements of the kernel of the Laplacian) of $EndE$ (which are in particular holomorphic), are exactly the diagonal endomorphisms with constants down the diagonal. In particular, if $\mathcal{E}$ is stable, then $\ker \Delta_{A^{EndE}}$ consists of precisely the endomorphisms $\mathbb{C} \cdot Id_E$.

Proof. We must show that a covariantly constant endomorphism is of the form

$$c_1 Id_{E_1} \oplus \cdots \oplus c_l Id_{E_l}.$$

Let $F \in EndE$ such that $d_{A^{End(E)}} F = 0$. This says in particular that F is holomorphic, which means that automatically the induced maps $F : \mathcal{E}_i \to \mathcal{E}_j$ are zero if $i < j$ (so that $\mu(\mathcal{E}_i) > \mu(\mathcal{E}_j)$) and $F = c_i Id_{E_i}$ if $i = j$, since $\mathcal{E}_i$ is stable, and in particular simple. We will write A_i for the induced connection on each bundle E_i. It is easy to check that $d_{A^{End(E)}} F = 0$ also implies that $d_{A^{Hom(E_i,E_j)}} F = 0$, where $A^{Hom(E_i,E_j)}$ is the connection on $Hom(E_i, E_j)$ induced by A_i and A_j. In other words, the induced map $F : \mathcal{E}_i \to \mathcal{E}_j$ is covariantly constant. We claim that such a map must vanish even if $\mu(\mathcal{E}_i) < \mu(\mathcal{E}_j)$. This is because the $\ker F \subset \mathcal{E}_i$ and $ImQ \subset \mathcal{E}_i$ will be a holomorphic sub-bundles and moreover will be invariant under d_{A_i} and d_{A_j} respectively, since the covariant constant condition says precisely that for any section σ of E_1,we have

$$d_{A_j}(F(\sigma)) = F(d_{A_i}(\sigma)).$$

It follows easily (see [KOB] Proposition 1.4.18) that $\mathcal{E}_i$ splits holomorphically as $\mathcal{E}_i = \ker F \oplus S$, for some holomorphic bundle S, and $\mathcal{E}_j$ as $\mathcal{E}_j = ImF \oplus Q$. If F is not the zero map, then we must have that $\ker F = 0$, since otherwise $\ker F$ and S are proper sub-bundles, contradicting the stability of $\mathcal{E}_i$. Similarly, since F is not zero, we must have that $Q = 0$ and $ImF = \mathcal{E}_j$ for the same reason. But these two conditions taken together imply that F is an isomorphism, which is impossible since $\mathcal{E}_i$ and $\mathcal{E}_j$ have different slopes. Therefore $F : \mathcal{E}_i \to \mathcal{E}_j$ must be zero whenever $i \neq j$, so we obtain the desired form for F. □

3.3. **The Yang-Mills flow on Riemann surfaces.** Recall that the holomorphic bundle $\mathcal{E}$ together with h gives the Chern connection A. We can produce a one parameter family $\mathcal{E}_t$ of holomorphic vector bundles associated to connections $A_t \in \mathcal{A}_h^{1,1}(E)$ given by the **Yang-Mills flow** starting at $A_0 = A$:

$$\begin{aligned} \frac{\partial A_t}{\partial t} &= -d^*_{A_t} F_{A_t}, \\ A_0 &= A. \end{aligned} \tag{3.10}$$

This equation is the gradient flow of the Yang-Mills functional. By Donaldson, it is known that the Yang-Mills flow has a global solution on $\mathcal{A}_h^{1,1} \times [0, \infty)$.

Using the Kahler identities, we can rewrite this equation as $\frac{\partial A_t}{\partial t} = d^{\mathbb{C}}_{A_t} \Lambda_\omega F_{A_t}$, where the operator $d^{\mathbb{C}}_{A_t} : \Omega^0(\mathfrak{u}(E)) \longrightarrow \Omega^1(\mathfrak{u}(E))$ is given by $d^{\mathbb{C}}_{A_t} = \sqrt{-1}(\bar{\partial}_{A_t} - \partial_{A_t})$. The tangent space to a $\mathcal{G}^{\mathbb{C}}$ orbit in $\mathcal{A}_h^{1,1}(E)$ at A_t is $imd_A \oplus imd^{\mathbb{C}}_A \subset \Omega^1(\mathfrak{u}(E)) = T_{A_t}\mathcal{A}_h^{1,1}(E)$, and therefore we see that the flow stays within a single complex gauge orbit.

A precise construction of the complex gauge transformations g_t which determine the Yang-Mills flow is as follows. If we assume that A_t is a solution to equation 3.10 and define g_t to be the unique solution to the ordinary differential equation defined by:

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= -(i\Lambda_\omega F_{A_t} \circ g_t - \mu_{\omega_\Sigma}(\mathcal{E})) \\ g_0 &= Id_E, \end{aligned} \tag{3.11}$$

then $g_t^*(A_0) = A_t$.

Note that

$$\begin{aligned} \bar{\partial}_{Hom(\mathcal{E},\mathcal{E}_t)}(g_t) &= \bar{\partial}_{A_t} \circ g_t - g_t \circ \bar{\partial}_A \\ &= g_t \circ \bar{\partial}_A - g_t \circ \bar{\partial}_A = 0, \end{aligned}$$

so $g_t : \mathcal{E} \to \mathcal{E}_t$ is a holomorphic map.

The flow 3.10 deforms a given connection in the direction of the gradient $d_A^* F_A$ of YM. By results of Uhlenbeck (see [U1], [U2]), any sequence of times along the flow converges to a Yang-Mills connection A_∞ on E, giving a holomorphic vector bundle $\mathcal{E}_\infty$. In general $\mathcal{E}_\infty$ is not isomorphic to $\mathcal{E}_0$, since by the previous discussion $\mathcal{E}_\infty$ must either be holomorphically split or stable. On the other hand, to any $\mathcal{E} \to (\Sigma, \omega)$, one can naturally associate a vector bundle (which topologically is the bundle E) whose holomorphic structure splits as a direct sum of stable bundles as follows.

Every such $\mathcal{E}$ admits a filtration by sub-bundles

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{l-1} \subset \mathcal{E}_l = \mathcal{E},$$

such that the successive quotients

$$\mathcal{Q}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$$

are slope stable. Such a filtration is obtained by combining the usual Harder-Narasiman filtration of a holomorphic bundle with a Jordan-Holder (or Seshadri) filtration of a semi-stable bundle. We will refer to this as a **Harder-Narasimhan-Seshadri filtration**. Although, this filtration is not quite unique, the point is that the associated graded object

$$Gr\,(\mathcal{E}) = \oplus_i \mathcal{Q}_i$$

is determined up to isomorphism entirely by $(\mathcal{E}, [\omega])$. Moreover, by the Donaldson-Uhlenbeck-Yau theorem, each $\mathcal{Q}_i$ admits an HYM connection, and their direct sum gives a Yang-Mills connection $A_{Gr(\mathcal{E})}$ on the bundle $Gr(\mathcal{E})$. This is therefore a natural candidate for the limit of the Yang-Mills flow. In fact the flow always converges to this connection.

Theorem 3.5. *(Daskalopoulos [D], Corollary 5.19 Råde [R], Prop 7.14) The Yang-Mills flow converges at infinity in the C^∞ topology in the space of connections $\mathcal{A}_h^{1,1}(E)$ to a Yang-Mills connection A_∞ giving rise to a holomorphic vector bundle $\mathcal{E}_\infty$, whose underlying smooth bundle is E. In fact $\mathcal{E}_\infty = Gr(\mathcal{E})$, and $A_\infty = A_{Gr(\mathcal{E})}$.*

Moreover, the flow converges at a rate of $1/\sqrt{t}$, that is, if $a_t \in \Omega^1(\mathfrak{u}(E))$ is defined by $a_t = A_t - A_\infty$, we have

$$\|a_t\|_{C^s} \leq C/\sqrt{t}$$

for all s and t sufficiently large.

Here, the statement that the limiting holomorphic structure is given by $Gr\,(\mathcal{E})$ is due to Daskalopoulos. The statement about the rate of convergence is due to Råde. Note that although the results of these papers give somewhat weaker convergence for the flow, this can be easily promoted to C^∞ convergence, see for example Section 3 of [W].

Lemma 3.6. *If A_t satisfies the Yang-Mills flow, then writing $A_t - A_\infty = a_t$ for a path $a_t \in \Omega^1(\mathfrak{u}(E))$. Then we have*

$$||\partial_t^j a_t||_{C^s} \leq C/\sqrt{t} \text{ and } ||\partial_t^j (\Lambda_{\omega_\Sigma} F_{A_t} - \Lambda_{\omega_\Sigma} F_{A_\infty})||_{C^s} \leq C/\sqrt{t},$$

for all j and s, and for t sufficiently large.

Proof. By the flow equations we have that

$$\begin{aligned}
\frac{\partial a_t}{\partial t} &= \sqrt{-1}\left(\overline{\partial}_{A_\infty} + a_t^{0,1} - \partial_{A_\infty} - a_t^{1,0}\right)(\Lambda_{\omega_\Sigma} F_{A_\infty} + \Lambda_{\omega_\Sigma} d_{A_\infty} a_t + \Lambda_{\omega_\Sigma} a_t \wedge a_t) \\
&= \sqrt{-1}\left(\overline{\partial}_{A_\infty}(\Lambda_{\omega_\Sigma} d_{A_\infty} a_t + \Lambda_{\omega_\Sigma} a_t \wedge a_t)\right) \\
&\quad + \sqrt{-1} a_t^{0,1} \wedge (\Lambda_{\omega_\Sigma} F_{A_\infty} + \Lambda_{\omega_\Sigma} d_{A_\infty} a_t + \Lambda_{\omega_\Sigma} a_t \wedge a_t) \\
&\quad - \sqrt{-1}\left(\partial_{A_\infty}(\Lambda_{\omega_\Sigma} d_{A_\infty} a_t - \Lambda_{\omega_\Sigma} a_t \wedge a_t)\right) \\
&\quad - \sqrt{-1} a_t^{1,0} \wedge ((\Lambda_{\omega_\Sigma} F_{A_\infty} + \Lambda_{\omega_\Sigma} d_{A_\infty} a_t + \Lambda_{\omega_\Sigma} a_t \wedge a_t))
\end{aligned}$$

where $a_t = a_t^{1,0} + a_t^{0,1}$, for $a_t^{1,0} \in \Omega^{1,0}(\mathfrak{u}(E))$, and $a_t^{0,1} \in \Omega^{0,1}(\mathfrak{u}(E))$. Note that the inner product on $\Omega^1(\mathfrak{u}(E))$ induced by g_Σ and h is orthogonal with respect to the decomposition $\Omega^1(\mathfrak{u}(E)) = \Omega^{1,0}(\mathfrak{u}(E)) \oplus \Omega^{0,1}(\mathfrak{u}(E))$, so

$$\left\|a_t^{1,0}\right\|_{C^s}^2, \left\|a_t^{0,1}\right\|_{C^s}^2 \leq \left\|a_t^{1,0}\right\|_{C^s}^2 + \left\|a_t^{0,1}\right\|_{C^s}^2 = \|a_t\|_{C^s}^2 \leq C/t$$

for t sufficiently large. Therefore

$$||\partial_t a_t||_{C^s} \leq C(\|a_t\|_{C^s} + \left\|a_t^{1,0}\right\|_{C^s} + \left\|a_t^{0,1}\right\|_{C^s}) \leq C\|a_t\|_{C^s} \leq C/\sqrt{t}.$$

Similarly, all derivatives of the expression for $\partial_t a_t$ will yield terms involving a_t, $a_t^{1,0}$, $a_t^{1,0}$ and higher time derivatives of these, so $||\partial_t^j a_t||_{C^s}$ can be bounded in the same way.

We also have

$$\Lambda_{\omega_\Sigma} F_{A_t} - \Lambda_{\omega_\Sigma} F_{A_\infty} = \Lambda_{\omega_\Sigma} d_{A_\infty} a_t + \Lambda_{\omega_\Sigma} a_t \wedge a_t,$$

so using the bound on $||\partial_t^j a_t||_{C^s}$, we obtain the same bound on $||\partial_t^j (\Lambda_{\omega_\Sigma} F_{A_t} - \Lambda_{\omega_\Sigma} F_{A_\infty})||_{C^s}$. □

3.4. **Hermitian-Yang-Mills flow.** In the above framework, the Hermitian bundle (E, h) remains fixed while the holomorphic structure moves. It will sometimes be useful to hold the complex structure on E defined by A_0 fixed, and instead move the Hermitian metric. In particular, we will let h evolve by the **Hermitian-Yang-Mills flow**

$$h_t^{-1}\frac{\partial h_t}{\partial t} = -2\left(i\Lambda_\omega F_{h_t} - \mu(\mathcal{E}) Id_E\right),$$

where F_{h_t} is the curvature of the Chern connection $A_{h_t} = (\bar{\partial}_A, h_t)$. Since we are assuming $\mu(\mathcal{E}) = 0$ (see the remark above) the equation becomes $h_t^{-1}\frac{\partial h_t}{\partial t} = -i\Lambda_\omega F_{h_t}$.

The Yang-Mills and Hermitian-Yang-Mills flow equations are equivalent up to gauge. If $A_t = g_t \cdot A_0$ is a solution of the Yang-Mills flow, then $h_t = h_0 g_t^* g_t$ is a solution of the Hermitian-Yang-Mills flow. Notice that h_t is by definition $g_t^{-1} \cdot h_0$. Conversely, if $h_t = h_0 k_t$ for a positive definite self-adjoint (with respect to h_0) endomorphism k_t, then $A_t = (k_t)^{\frac{1}{2}} A_0$ is real gauge equivalent to a solution of the Yang-Mills flow. To spell out the equivalence precisely, the map:

$$g_t : (\mathcal{E}, h_0 k_t) \longrightarrow (\mathcal{E}_t, h_0)$$

is a biholomorphism and an isometry, where $k_t = g_t^* g_t$. Therefore, since the YM flow exist for all time, so does the HYM flow.

The following calculation gives the relationship between the curvature of the connection A_t, and that of A_{h_t}. The action of g_t on a connection $\nabla_A = \partial_{A,h} + \bar{\partial}_A \in \mathcal{A}_h^{1,1}(E)$, is

$$\nabla_{A_t} = g_t \cdot \nabla_A = (g_t^*)^{-1} \circ \partial_A \circ (g_t^*) + g_t \circ \bar{\partial}_A \circ g_t^{-1}.$$

Notice that g_t is not metric preserving, so this is different from the action of the group $\mathcal{G}$. Conjugating by g_t^{-1} we have

$$g_t^{-1} \circ \nabla_{A_t} \circ g_t = k_t^{-1} \circ \partial_A \circ k_t + \bar{\partial}_A$$

where $k_t = g_t^* g_t$. The connection $k_t^{-1} \circ \partial_A \circ k_t + \bar{\partial}$ is precisely the Chern connection $(\bar{\partial}_A, h_t)$. Therefore, composing the above formula with itself we have that

$$g_t^{-1} \circ F_{A_t} \circ g_t = F_{h_t}.$$

3.5. **Examples.** In this section we give examples where the conditions of the main theorem hold.

Example 3.7. We recall the example sketched in the introduction. Namely, consider a rank two bundle $\mathcal{E} \to \Sigma$ which is given as a non-split extension

$$0 \to \mathcal{L}_1 \xrightarrow{\imath} \mathcal{E} \xrightarrow{\pi} \mathcal{L}_2 \to 0,$$

where $\mu_{\omega_\Sigma}(\mathcal{L}_1) > \mu_{\omega_\Sigma}(\mathcal{L}_2)$. By construction, this sequence is precisely the Harder-Narasimhan filtration of $\mathcal{E}$, since the condition on the slope implies that $\mathcal{L}_1$ destabilises $\mathcal{E}$. As $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles they are stable, and thus $Gr(\mathcal{E}) = \mathcal{L}_1 \oplus \mathcal{L}_2$ satisfies the condition stated in Theorem 1.3. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ also satisfy the conditions $Hom(\mathcal{L}_2, \mathcal{L}_1) = 0$. We claim that $\mathcal{E}$ must also be simple. Applying $Hom(-, \mathcal{E})$, we obtain an exact sequence

$$0 \to Hom(\mathcal{L}_2, \mathcal{E}) \xrightarrow{\pi^*} Hom(\mathcal{E}, \mathcal{E}) \xrightarrow{\imath^*} Hom(\mathcal{L}_1, \mathcal{E}).$$

We claim that the first map is 0. then we must have $im(f) \subset \ker(\pi) = \mathcal{L}_1$, since otherwise $\pi \circ f$ gives a non-trivial map $\mathcal{L}_2 \to \mathcal{L}_2$, and since $\mathcal{L}_2$ is stable this map must be a constant multiple of $Id_{\mathcal{L}_2}$, giving a splitting of the sequence. Then by assumption $Hom(\mathcal{L}_2, \mathcal{E}) = Hom(\mathcal{L}_2, \mathcal{L}_1) = 0$, and so we obtain an injection $Hom(\mathcal{E}, \mathcal{E}) \hookrightarrow Hom(\mathcal{L}_1, \mathcal{E})$. On the other hand, any map $f : \mathcal{L}_1 \to \mathcal{E}$, must have image contained in $\mathcal{L}_1$, since otherwise $\pi \circ f : \mathcal{L}_1 \to \mathcal{L}_2$ gives a non-trivial map, which is impossible by the condition on the slopes. Therefore we have $Hom(\mathcal{E}, \mathcal{E}) \simeq Hom(\mathcal{L}_1, \mathcal{L}_1) = \mathbb{C}$, since $\mathcal{L}_1$ is stable. Note that the extensions of $\mathcal{L}_1$ by $\mathcal{L}_2$ are classified by $H^1(\mathcal{L}_2^* \otimes \mathcal{L}_1)$. Therefore it suffices to find line bundles satisying the condition $H^0(\mathcal{L}_2^* \otimes \mathcal{L}_1) = 0$ and $\dim H^1(\mathcal{L}_2^* \otimes \mathcal{L}_1) > 0$. As in [HA], we may find line bundles with $\deg(\mathcal{L}_1) = 1$ and $\deg(\mathcal{L}_2) = 0$ satisfying the first condition, as $\deg(\mathcal{L}_2^* \otimes \mathcal{L}_1) = 0$, and there are plenty of non-effective divisors of degree 1. If we assume $g(\Sigma) = 3$, then the Riemann-Roch theorem gives $\dim H^1(\mathcal{L}_2^* \otimes \mathcal{L}_1) = g - 1 - \deg(\mathcal{L}_1) = 1$ so we obtain explicit examples as soon as $g = 3$. In fact, the Picard groups of line bundles of degrees 1 and 0 are both three dimensional, and we have only used the open condition $\dim H^1(\mathcal{L}_2^* \otimes \mathcal{L}_1) > 0$, so there is a six dimensional family of such extensions.

4. Some background on projective bundles

4.1. **Projective bundles, connections, and holomorphic structures.** We fix the bundle $(E, h) \to (\Sigma, \omega_\Sigma)$ as in the last section. We will study the projectivisation $\mathbb{P}(E)$ which is a smooth manifold equipped with a natural smooth projection map $\pi : \mathbb{P}(E) \to \Sigma$. There is also a projectivisation map $\xi : E \to \mathbb{P}(E)$ that takes a vector $v \in E$ to its projective equivalence class $[v]$. By construction we have $\tilde{\pi} = \pi \circ \xi$.

Now fix a connection $\nabla_A \in \mathcal{A}_h^{1,1}(E)$ on (E, h) giving a the holomorphic bundle $\mathcal{E}$ as in the previous section. Then $\mathbb{P}(\mathcal{E})$ is a complex manifold and the map $\pi : \mathbb{P}(\mathcal{E}) \to \Sigma$ is holomorphic. In particular ∇_A determines an integrable almost complex structure on $\mathbb{P}(E)$ denoted by J, which we will now describe. Recall that the vertical bundle is the fixed smooth subbundle V_E given by the kernel of the map $d\tilde{\pi} : TE \to T\Sigma$ The choice of connection ∇_A is tantamont to a choice of horizontal complementary subbundle $H_E^A \subset TE$ (with respect to g_Σ) so that we obtain a splitting

$$TE = V_E \oplus H_E^A. \tag{4.1}$$

Since each fibre of E is a vector space, for any $v \in E$ with $\tilde{\pi}(v) = x$ for the tangent space to the fibre we have $T_v E_x \cong E_x$ and in fact there is a global isomorphism $V_E \cong \tilde{\pi}^* E$. The map $d\tilde{\pi}$ then gives a smooth isomorphism $H_E^A \cong \tilde{\pi}^*(T\Sigma)$ for any connection A. The horizontal-vertical splitting then gives an isomorphism

$$TE \cong \tilde{\pi}^*(E) \oplus \tilde{\pi}^*(T\Sigma).$$

We may define an almost complex structure $\tilde{J}_A$ on the total space of E by pulling back the direct sum

$$j_E \oplus J_\Sigma$$

under this isomorphism, where j_E is multiplication by i on the fibres, and J_Σ is the (integrable) almost complex structure on Σ. The integrability condition on A can be used to show that this almost complex structure is integrable, and so defines the structure of a complex manifold on E (and therefore that of a holomorphic vector bundle). By the chain rule we have $d\tilde{\pi} = d\pi \circ d\xi$, therefore since ξ is a submersion, $d\xi$ restricts to H_E^A to an isomorphism onto its image, which we denote by H_A, and gives a surjection of V_E onto $V = \ker d\pi$. We therefore obtain a smooth splitting

$$T\mathbb{P}(E) = V \oplus H_A. \tag{4.2}$$

Since $v \in \ker d\xi$ if and only if $j_E(v) = iv \in \ker d\xi$, the almost complex structure given above descends to $\mathbb{P}(E)$ to give an integrable almost complex structure J_A.

Then J_A gives the complex structure associated to $\mathbb{P}(\mathcal{E})$, which could also be obtained from the holomorphic charts for $\mathcal{E}$. Notice that the d-bar operator on functions, which will appear throughout the rest of the paper may be defined by

$$\overline{\partial}_{J_A} = d - iJ_A d. \tag{4.3}$$

In the sequel we will simply denote these operators by J, and $\overline{\partial}_J$ whenever the connection on E is fixed.

Finally since $d\pi$ is actually a map holomorphic bundles, the bundle V inherits a holomorphic structure from $T\mathbb{P}(\mathcal{E})$. We write $\mathcal{V}$ for the resulting holomorphic bundle, and writing $\mathcal{H}$ for the quotient, we obtain an exact sequence of holomorphic bundles

$$0 \longrightarrow \mathcal{V} \longrightarrow T\mathbb{P}(\mathcal{E}) \longrightarrow \mathcal{H} \longrightarrow 0 \tag{4.4}$$

where $\mathcal{H}$ is isomorphic as a smooth bundle to H_A. In the sequel we will write H for this latter bundle if a fixed connection is understood.

4.2. **Gauge diffeomorphisms and moving holomorphic structures.** The gauge transformations g_t introduced in the last section induce diffeomorphisms $\tilde{g}_t : \mathbb{P}(E) \to \mathbb{P}(E)$ by $\tilde{g}_t(x, [v]) = (x, [\tilde{g}_t(v)])$. For the moving holomorphic structure J_t, associated with the connection A_t, we have an associated operator $\bar{\partial}_{J_t}$ of smooth functions of $\mathbb{P}(E)$, and its conjugate ∂_{J_t}, so that for each t we have $d = \bar{\partial}_{J_t} + \partial_{J_t}$. Since $\tilde{g}_t : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}_t)$ is holomorphic we have that

$$\bar{\partial}_J \circ \tilde{g}_t{}^* = \tilde{g}_t^* \circ \bar{\partial}_{J_t}$$

so that

$$\bar{\partial}_{J_t} = (\tilde{g}_t^{-1})^* \circ \bar{\partial}_J \circ \tilde{g}_t{}^*.$$

Then we can write $\tilde{g}_t^* \circ \bar{\partial}_{J_t} + \tilde{g}_t^* \circ \partial_{J_t} = \tilde{g}_t^* \circ d = d \circ \tilde{g}_t^* = \partial_J \circ \tilde{g}_t^* + \bar{\partial}_J \circ \tilde{g}_t^*$ which implies that

$$\partial_{J_t} = (\tilde{g}_t^{-1})^* \circ \partial_J \circ \tilde{g}_t^*.$$

4.3. **The hyperplane bundle and its curvature.** Given a holomorphic vector bundle $\mathcal{E}$ of rank r, and its projectivisation $\mathbb{P}(\mathcal{E})$, recall the that there is holomorphic line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathbb{P}(\mathcal{E})$, which is the line sub-bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \subset \pi^*\mathcal{E}$ defined fibrewise by the usual tautological line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$.

Then notice then we have an exact sequence of holomorphic bundles

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0,$$

and therefore $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ inherits a metric $h_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)}$ Locally we may write

$$iF_{h_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)}} = i\bar{\partial}\partial \text{log} h.$$

If h_1 and h_2 are two different hermitian metrics on E, then if we define the smooth function f on $\mathbb{P}(E)$ by

$$f([v]) = \log\left(\frac{h_1(v,v)}{h_2(v,v)}\right),$$

then the curvatures of the Chern connections of satisfy $iF_{(h_1, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))} = iF_{(h_2, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))} + i\bar{\partial}\partial f$.

The dual of this metric gives a metric $h_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)}$ on the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)}$, and if f is defined above then

$$iF_{(h_1, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} = iF_{(h_2, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} - i\bar{\partial}\partial f. \tag{4.5}$$

By Chern-Weil theory the cohomology class $2\pi c_1\left(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right)$ is represented by $iF_{(h, \mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}$.

4.4. **The moment map, the Fubini-Study form, and the decomposition of the curvature.** We will see that $iF_{(h, \mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}$ on $\mathbb{P}(E)$ naturally decomposes into two pieces. The splitting 4.2 yields a decomposition:

$$\begin{aligned} \Lambda^2(T^*\mathbb{P}(E)) &= \Lambda^2(V^*) \oplus (V^* \otimes H) \oplus \Lambda^2(H^*) \\ &= \Lambda^2(V^*) \oplus (V^* \otimes H) \oplus (\Lambda^2\pi^*(T^*X)). \end{aligned}$$

This means that for $F \in \Omega^2(\mathbb{P}(E))$ we may write $F = F_{HH} + F_{HV} + F_{VV}$. In particular curvature $iF_{(h, \mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})} \in \Omega^2(\mathbb{P}(E))$ has such a decomposition, and we will need to understand this more precisely.

We define a map

$$\Phi_h : End(E) \to C^\infty(\mathbb{P}(E))$$

by

$$\Phi_h(F)([v]) = i\frac{h_{\pi([v])}(Fv, v)}{\|v\|_h^2}. \tag{4.6}$$

Note that since $End(E) = \mathfrak{u}(E,h) \oplus i\mathfrak{u}(E,h)$, this also defines maps

$$\Phi_h : \Gamma(\mathfrak{u}(E,h)) \to C^\infty(\mathbb{P}(E)), \Phi_h : \Gamma(i\mathfrak{u}(E,h)) \to C^\infty(\mathbb{P}(E))$$

from the hermitan and skew-hermitian matrices. If we combine Φ_h with the pullback map $\pi^* : \Omega^k(\Sigma) \to \Omega^k(\mathbb{P}(E))$, we obtain a map

$$\Phi_h : \Omega^k(\mathfrak{u}(E,h)) \to \Omega^k(\mathbb{P}(E)).$$

In practice we will mostly be concerned with the case where F is the curvature F_A of a connection A. Since Σ is a Riemann surface, notice that that $F_A = (\Lambda_{\omega_\Sigma} F_A)\,\omega_\Sigma$, so

$$\Phi_h(F_A) = \Phi_h(\Lambda_{\omega_\Sigma} F_A)\omega_\Sigma. \tag{4.7}$$

The Hermitian metric h on E defines a fundamental form $\Omega(h) = \frac{i}{2}(h - \overline{h})$ (or strictly speaking the pullback of this formula to E), which we thing of as a vertical $(1,1)$ form on E. If we write $\imath_{\mathbb{S}} : \mathbb{S}(E) \hookrightarrow E$ for the inclusion of the sphere bundle, then we may define a two form $\omega_{FS}(h)$ on $\mathbb{P}(E)$ which is characterised by the formula

$$\xi^*(\omega_{FS}(h)) = \imath_{\mathbb{S}}^*(\Omega(h)), \tag{4.8}$$

where we recall that $\xi : \mathbb{S}(E) \to \mathbb{P}(E)$ is the projectivisation map. Since $\Omega(h)$ is vertical, so is $\omega_{FS}(h)$. By construction, $\omega_{FS}(h)$ restricted to a fibre is the Fubini-Study metric associated to the restriction of h to the corresponding fibre of E.

The following is Formula 15.15 of [DEM].

Lemma 4.1. *With respect to the above splitting the two form $iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}$ decomposes as*

$$iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})} = \omega_{FS}(h) + \Phi_h\left(-F_A\right) \in \Gamma\left(\Lambda^2(V^*)\right) \oplus \Gamma\left(\Lambda^2(H^*)\right),$$

where $\Phi_h\left(-F_A\right)$ is as above, and $\omega_{FS}(h)$ is the vertical 2-form that restricts to each fibre to be the Fubini-Study form. In other words, $\left(iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}\right)_{HH} = \Phi_h\left(-F_A\right)$, $\left(iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}\right)_{HV} = 0$, and $\left(iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})}\right)_{VV} = \omega_{FS}(h)$.

4.5. **Kähler metrics on $\mathbb{P}(\mathcal{E})$, $\mathbb{P}(\mathcal{E}_t)$ and $\mathbb{P}(\mathcal{E}_\infty)$.** For any metric h on E, and any holomorphic structure $\overline{\partial}_E$ with Chern connection $\nabla_A = (\overline{\partial}_E, h)$, inducing a complex structure J on $\mathbb{P}(\mathcal{E})$, and for any positive integer k, we will define the two-form

$$\begin{aligned} \omega_k(h,J) &= iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})(1)})} + k\pi^*\omega_\Sigma \\ &= \omega_{FS}(h) + (\Phi_h(-\Lambda_{\omega_\Sigma} F_A) + k)\,\omega_\Sigma. \end{aligned} \tag{4.9}$$

Notice that since $\omega_{FS}(h)$ is positive on the vertical sub-bundle $\mathcal{V} \subset T\mathbb{P}(\mathcal{E})$ (and 0 on the horizontal sub-bundle), if $k > \inf \Phi_h(-\Lambda_{\omega_\Sigma} F_A)$, this two-form is positive definite, and therefore $\omega_k(h,J)$ defines a Kähler metric.

We will write J_t for the holomorphic structure on $\mathbb{P}(\mathcal{E}_t)$. For each t we have a hyperplane bundle $\mathcal{L}_t = \mathcal{O}_{\mathbb{P}(\mathcal{E}_t)}(1) \to \mathbb{P}(\mathcal{E}_t)$. and again h induces a metric on this line bundle whose curvature gives a closed two form on $\mathbb{P}(\mathcal{E}_t)$ compatible with the holomorphic structure $\bar{\partial}_{\mathcal{E}_t}$. Throughout this section, we fix an Hermitian metric h on E. Let $V \to \mathbb{P}(E)$ be the vertical sub-bundle of $T\mathbb{P}(E)$ with fibre $V_{[z]} = T_{[z]}\mathbb{P}(E_x)$, where $\pi([z]) = x$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{V}_t \longrightarrow T\mathbb{P}(\mathcal{E}_t) \longrightarrow \mathcal{H}_t \longrightarrow 0$$

where $\mathcal{H}_t$ is by definition the holomorphic vector bundle given by the quotient, which is smoothly isomorphic to the complementary subbundle $H_t \subset T\mathbb{P}(E)$ to V determined by the connection A_t.

Then for each k we also have a one parameter family

$$\begin{aligned} \omega_k\left(h,J_t\right) &= \omega(h,J_t) + k\omega_\Sigma \\ &= (\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\,\omega_\Sigma + k\omega_{FS}(h). \end{aligned} \tag{4.10}$$

for each t the $\omega_k\left(h,J_t\right)$ are compatible with the complex structure J_t, for all t. In other words for sufficiently large k, $\omega_k\left(h,J_t\right)$ is a Kähler form on the complex manifold $\mathbb{P}(\mathcal{E}_t)$.

In the same way we define a path of Kähler metrics on the fixed complex manifold $\mathbb{P}(\mathcal{E})$ by applying this construction to the family of Hermitian metrics given by the HYM flow. Namely define:

$$\begin{aligned} \omega_k\,(h_t, J) &= \omega(h_t, J) + k\omega_\Sigma \\ &= \left(\Phi_{h_t}(-\Lambda_{\omega_\Sigma} F_{h_t}) + k\right)\omega_\Sigma + \omega_{FS}(h_t). \end{aligned} \tag{4.11}$$

Lemma 4.2. *We have*

$$\omega(h_t, J) = \widetilde{g}_t^*(\omega(h, J_t)).$$

Proof. By Lemma 4.12 below we have

$$\widetilde{g}_t^*\left(\Phi_h\left(\Lambda_{\omega_\Sigma} F_{A_t}\right) + k)\omega_\Sigma\right) = \Phi_{h_t}\left(\Lambda_{\omega_\Sigma} F_{h_t}\right) + k)\omega_\Sigma.$$

For the vertical part, notice that

$$\imath_{\mathbb{S}}^* \circ g_t^*(\imath_{\mathbb{S}}^*\Omega(h)) = \frac{i}{2}\imath_{\mathbb{S}}^* \circ g_t^*(h - \overline{h})|_{\mathbb{S}(E)} = \frac{i}{2}(h_t - \overline{h}_t)|_{\mathbb{S}(E)} = \imath_{\mathbb{S}}^*\Omega(h_t).$$

Therefore

$$\begin{aligned} \imath_{\mathbb{S}}^*\Omega(h_t) &= \imath_{\mathbb{S}}^* \circ g_t^*\left(\imath_{\mathbb{S}}^*\Omega(h)\right) = \imath_{\mathbb{S}}^* \circ g_t^*\left(\xi^*(\omega_{FS}(h))\right) \\ &= \imath_{\mathbb{S}}^* \circ (\xi \circ g_t)^*\,\omega_{FS}(h) = \imath_{\mathbb{S}}^* \circ (\widetilde{g}_t \circ \xi)^*\,\omega_{FS}(h) \\ &= \imath_{\mathbb{S}}^* \circ \xi^*\left(\widetilde{g}_t^*(\omega_{FS}(h)\right) = (\xi \circ \imath_{\mathbb{S}})^*\left(\widetilde{g}_t^*(\omega_{FS}(h)\right), \end{aligned}$$

which implies

$$\omega_{FS}(h_t) = \widetilde{g}_t^*(\omega_{FS}(h)).$$

Combining these two equalities, we obtain the result. □

Consider the the limiting holomorphic structure J_∞ corresponding to the holomorphic structure induced by the limiting connection A_∞ of the Yang-Mills flow on E, giving the complex manifold $\mathbb{P}(E, J_\infty) = \mathbb{P}(Gr(\mathcal{E}))$. Then we may consider the two form $\omega(h, J_\infty) = iF_{\nabla_{\mathcal{L}_\infty}(h_{\mathcal{L}_\infty})}$, where $\nabla_{\mathcal{L}_\infty}(h_{\mathcal{L}_\infty})$ is the Chern connection on the line bundle $\mathcal{L}_\infty = \mathcal{O}_{\mathbb{P}(\mathcal{E}_\infty)}(1) \to \mathbb{P}(\mathcal{E}_\infty)$, with the metric $h_{\mathcal{L}_\infty}$ induced on $\mathcal{L}_\infty$ by h. This gives a Kähler metric

$$\omega_k(h, J_\infty) = \omega(h, J_\infty) + k\pi^*\omega_\Sigma \tag{4.12}$$

on the manifold $\mathbb{P}(\mathcal{E}_\infty)$. We will write $g_{k,\infty}$ for the associated Riemannian metric on the smooth manifold $\mathbb{P}(E)$.

4.6. **Vector fields on $\mathbb{P}(\mathcal{E})$, $\mathbb{P}(\mathcal{E}_t)$, and $\mathbb{P}(\mathcal{E}_\infty)$.** We begin by giving a construction of smooth vertical vector fields on $\mathbb{P}(E)$. An endomorphism $F : E \to E$, defines a vertical vector field $\tilde{X}_F$ by

$$\tilde{X}_F\,(v) = F(v),$$

where $v \in E_{\tilde{\pi}(v)}$ and where we are using the isomorphism $T_v E_{\tilde{\pi}(v)} \cong E_{\tilde{\pi}(v)}$.

The vector field $\tilde{X}_F$ descends to a vertical vector field X_F on $\mathbb{P}(E)$ as follows. Recall that the fibres $T[v]\mathbb{P}(E_x)$ of the bundle $V \subset T\mathbb{P}(E)$ may be identified with the space $Hom([v], E_x/[v])$. Under this identification the differential $d\xi$ is identified with the map taking a vector $w \in T_v E$, to the endomorphism

$$\lambda v \mapsto \lambda proj_{[v]}^{\perp}(w) := \lambda\left(w - \frac{h(w, v)}{h(v, v)}v\right).$$

We define the value of the vector field X_F at a point $[v]$ similarly to be the endomorphism

$$X_F\left([v]\right) : \lambda v \to \lambda \left(Fv - \frac{h_{\pi([v])}(Fv, v)}{\|v\|_h^2} v \right). \tag{4.13}$$

As a consequence we may also write:

$$X_F = d\xi \left(\widetilde{X}_F \right),$$

which by the formula for the derivative is unambiguous. Notice however, that the formula for X_F depends on the choice of metric h. When we need to emphasise the metric we will write X_F^h for this vector field, but otherwise we will omit the h.

The following lemma will be crucial to our application of the inverse function theorem later on.

Lemma 4.3. *Let $\mathcal{E} \longrightarrow \Sigma$ be a simple bundle. Then $\mathbb{P}(\mathcal{E})$ has no holomorphic vector fields if $g(\Sigma) \geq 2$. If $g(\Sigma) = 1$, then $T\Sigma$ is trivial and the only holomorphic vector fields on $\mathbb{P}(\mathcal{E})$ are pullbacks of the constant vector fields on Σ. Since the Yang-Mills flow stays inside of a single complex gauge orbit, this remains true for the bundles $\mathcal{E}_t$ determined by the flow.*

Proof. The usual short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow T\mathbb{P}(\mathcal{E}) \longrightarrow \pi^*\left(T\Sigma\right) \longrightarrow 0,$$

gives a long exact sequence in cohomology of the form

$$0 \to H^0\left(\mathcal{V}\right) \to H^0\left(T\mathbb{P}(\mathcal{E})\right) \to H^0\left(\pi^*\left(T\Sigma\right)\right) \to \cdots$$

Then either $H^0\left(\pi^*\left(T\Sigma\right)\right) = H^0\left(T\Sigma\right) = 0$ (if $g \geq 2$), or $H^0\left(\pi^*\left(T\Sigma\right)\right) = H^0\left(T\Sigma\right) = \mathbb{C}$ (if $g = 1$). In the former case we obtain

$$H^0\left(T\mathbb{P}(\mathcal{E})\right) \simeq H^0\left(\mathcal{V}\right),$$

and in the latter case we have a splitting

$$H^0\left(T\mathbb{P}(\mathcal{E})\right) \simeq H^0\left(\mathcal{V}\right) \oplus \mathbb{C}.$$

We may identify$H^0(\mathcal{V})$ with the traceless endomorphisms, that is sections $H^0(\mathcal{E}nd_0\left(\mathcal{E}\right)) = 0$ (since $\mathcal{E}$ is simple), as follows. The globalisation of the the Euler sequence on the fibres to $\mathbb{P}(\mathcal{E})$ is given by

$$0 \longrightarrow \underline{\mathbb{C}} \to \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \longrightarrow \mathcal{V} \longrightarrow 0.$$

Taking the pushforward of this sequence and using the push-pull formula, and the fact that $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq S^1\mathcal{E}^* = \mathcal{E}^*$, we obtain an exact sequence on Σ :

$$0 \longrightarrow \underline{\mathbb{C}} \to \mathcal{E} \otimes \mathcal{E}^* = \mathcal{E}nd(\mathcal{E}) \longrightarrow \pi_*\mathcal{V} \longrightarrow 0.$$

The long exact sequence in cohomology then gives

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(\mathcal{E}nd(\mathcal{E})) \longrightarrow H^0(\pi_*\mathcal{V}) = H^0(\mathcal{V}) \longrightarrow 0,$$

where we have also used the definition of the pushforward. The map $H^0(\mathcal{E}nd(\mathcal{E})) \to H^0(\mathcal{V})$ may be thought of as the map $F \mapsto (X_F)^{1,0}$, whose kernel may be identified with the constant multiples of the identity on $\mathcal{E}$. We therefore obtain an isomorphism

$$H^0(\mathcal{E}nd_0(\mathcal{E})) \simeq H^0(\mathcal{E}nd_0(\mathcal{E}))/\mathbb{C} \simeq H^0(\mathcal{V}).$$

Then we have either:

$$\begin{aligned} H^0\left(T\mathbb{P}(\mathcal{E})\right) &= 0, \\ \text{or } H^0\left(T\mathbb{P}(\mathcal{E})\right) &= \mathbb{C}, \end{aligned}$$

according to the genus. □

Corollary 4.4. *If $\mathcal{E}$ is simple, for any Kähler metric ω on $\mathbb{P}(\mathcal{E})$ (and in particular for $\omega_k(h,J)$) we have*

$$\ker \mathfrak{D}^*_\omega \mathfrak{D}_\omega \simeq \mathbb{C}.$$

Proof. This follows directly form Equation 2.4 and Lemma 4.3. □

Lemma 4.5. *If $A \in \mathcal{A}^{1,1}_h(E)$ and J is the holomorphic structure on $\mathbb{P}(E)$ corresponding to $\overline{\partial}_A$, with corresponding horizontal and vertical bundles $\mathcal{H}$ and $\mathcal{V}$, then for $F \in \Gamma\left(End(E)\right)$ we have*

$$\left(d\Phi_h\left(F\right)\right)_{\mathcal{H}} = \Phi_h\left(d_A F\right).$$

If $F_1 \in \Gamma(\mathfrak{u}(E))$, $F_2 \in \Gamma(i\mathfrak{u}(E))$ we have,

$$\left(d\Phi_h\left(F_1\right)\right)_{\mathcal{V}} = \omega_{FS}(h)\left(X_{F_1}, -\right), \left(d\Phi_h\left(F_2\right)\right)_{\mathcal{V}} = -i\omega_{FS}(h)\left(JX_{F_2}, -\right).$$

As a consequence, if $F = F_1 + F_2$ is covariantly constant with respect to A, then

$$d\Phi_h\left(F\right) = \omega_{FS}(h)\left(X_F, -\right) - i\omega_{FS}(h)\left(JX_{F_2}, -\right).$$

In particular if A is Yang-Mills,

$$d\Phi_h\left(\Lambda_{\omega_\Sigma} F_A\right) = \omega_{FS}(h)\left(X_{\Lambda_{\omega_\Sigma} F_A}, -\right).$$

Proof. By definition, for any point $x \in \Sigma$, and any v with $\widetilde{\pi}(v) = x$, $H_v \subset T_v E$, is defined by $H_v = d\sigma_x(T_x\Sigma)$ for some section σ of E defined locally near x for which $\sigma(x) = v$ and $(\nabla\sigma)_x = 0$. Then for such a choice of x, v, and σ, and a vector field $X_\Sigma \in \Gamma\left(T\Sigma\right)$, and defining $d\sigma(X_\Sigma) = X = X_H + X_V$ we may write

$$\begin{aligned}
d(\widetilde{\Phi}_h(F) \circ \sigma)_x\left(X_x\right) &= \left(d\widetilde{\Phi}_h(F)\right)_v \circ d\sigma_x\left((X_\Sigma)_x\right) \\
&= \left(d\widetilde{\Phi}_h(F)\right)_v \left((X_H)_v + (\widehat{\nabla^X\sigma})_v\right) \\
&= \left(d\widetilde{\Phi}_h(F)\right)_v \left((X_H)_v + (\nabla^X\sigma)_x\right) \\
&= \left(d(\widetilde{\Phi}_h(F))_H\right)_v \left((X_H)_v\right),
\end{aligned}$$

where we have used the hat notation to again to denote the pullback of the section $\nabla^X\sigma$ and the basic fact that this section (thought of as a vertical vector field) is precisely the vertical component of $d\sigma$; as well as the defining condition for σ at x. On the other hand since A is an hermitian connection

$$\begin{aligned}
d(\widetilde{\Phi}_h(F) \circ \sigma)_x\left(X_x\right) &= \left(idh(F(\sigma), \sigma)\right)_x\left((X_\Sigma)_x\right) \\
&= \left(ih(d_A(F(\sigma)(X_\Sigma)), \sigma) - ih(F(\sigma), \nabla_A^{X_\Sigma}\sigma)\right)_x \\
&= \left(ih(d_A(F)(\sigma)(X_\Sigma) - F \circ \nabla_A^{X_\Sigma}\sigma, \sigma) - ih(F(\sigma), \nabla_A^{X_\Sigma}\sigma)\right)_x \\
&= ih(d_A(F)(\sigma), \sigma)_x\left(X_\Sigma\right)_x = \widetilde{\Phi}_h(d_A F\left(X_\Sigma\right)_x))_{(\sigma(x))} \\
&= \widetilde{\Phi}_h(d_A F))_{(\sigma(x))}\left((X_H)_v\right).
\end{aligned}$$

Since v was arbitrary we obtain

$$d(\widetilde{\Phi}_h(F))_H = \widetilde{\Phi}_h(d_A F)).$$

On the other hand, by construction $\widetilde{\Phi}_h\left(F\right) = \Phi_h\left(F\right) \circ \xi$ so

$$\Phi_h(d_A F)) = \widetilde{\Phi}_h(d_A F)) \circ \xi = \xi^* d(\widetilde{\Phi}_h(F))_H$$

$$= \quad d(\xi^*\widetilde{\Phi}_h(F))_H = d(\Phi_h(F))_H,$$

and we obtain the first result. The other results amount to the statement that the restriction of $\Phi_h(F)$ to the fibres is a moment map. $\square$

Corollary 4.6. *If $A \in \mathcal{A}_h^{1,1}(E)$ and J is the holomorphic structure on $\mathbb{P}(E)$ corresponding to $\overline{\partial}_A$, for $F_1 \in \mathfrak{u}(E)$, $F_2 \in \Gamma(i\mathfrak{u}(E))$ we have*

$$X_{F_1} = J\left(\nabla_{g_{k(h,J)}}\Phi_h(-F)\right)_{\mathcal{V}}, \; iX_{F_2} = \left(\nabla_{g_{k(h,J)}}\Phi_h(F)\right)_{\mathcal{V}}$$

If $F = F_1 + F_2$ is covariantly constant with respect to A, then

$$X_{F_1} - iJX_{F_2} = J\nabla_{g_{k(h,J)}}\Phi_h(-F).$$

In particular, if A is Yang-Mills, then

$$X_{\Lambda_{\omega_\Sigma}F_A} = J\nabla_{g_{k(h,J)}}\Phi_h(-\Lambda_{\omega_\Sigma}F_A).$$

Proof. By the previous lemma, for $F_1 \in \Gamma(\mathfrak{u}(E))$, $F_2 \in \Gamma(i\mathfrak{u}(E))$ we have

$$\begin{aligned}(d\Phi_h(F_1))_{\mathcal{V}} &= \omega_{FS}(h)(X_{F_1}, -) = g_{FS}(h)(JX_{F_1}, -),\\ (d\Phi_h(F_2))_{\mathcal{V}} &= -i\omega_{FS}(h)(JX_{F_2}, -) = g_{FS}(h)(X_{F_2}, -),\end{aligned}$$

so we must have

$$JX_{F_1} = \left(\nabla_{g_{k(h,J)}}\Phi_h(F_1)\right)_{\mathcal{V}}, \; X_{F_1} = \left(\nabla_{g_{k(h,J)}}\Phi_h(F_1)\right)_{\mathcal{V}},$$

which gives the first result. If F is covariantly constant with respect to A, then again by the previous lemma we obtain,

$$\begin{aligned}d\Phi_h(F_1) &= \omega_{FS}(h)(X_{F_1}, -) + \Phi_h(d_AF_1) = g_{FS}(h)(JX_F, -) + \Phi_h(d_AF_1)\\ &= g_{k(h,J)}((JX_{F_1}, -)) + \Phi_h(d_AF_1),\\ &= g_{k(h,J)}((JX_{F_1}, -))\\ d\Phi_h(F_2) &= ig_{k(h,J)}((X_{F_2}, -)) + \Phi_h(d_AF_2)\\ &= ig_{k(h,J)}((X_{F_2}, -)),\end{aligned}$$

so that

$$d\Phi_h(F) = g_{k(h,J)}((JX_{F_1} + iX_{F_2}, -))$$
$$JX_{F_1} + iX_{F_2} = \nabla_{g_{k(h,J)}}\Phi_h(F)$$

giving the second result. If A is Yang-Mills then by equation 3.8, $\Lambda_{\omega_\Sigma}F_A$ is covariantly constant with respect to A, so we obtain the final result. $\square$

Lemma 4.7. *Let $\mathcal{E}$ be a holomorphic vector bundle with underlying smooth bundle E, and $F \in \Gamma(EndE)$. If $F \in H^0(\mathcal{E}nd(\mathcal{E}))$ is a holomorphic endomorphism, the vector field X_F is real holomorphic. That is, $(X_F)^{1,0} \in T^{1,0}(\mathbb{P}(\mathcal{E}))$ is a holomorphic vector field.*

Proof. Recall that X_F is the image under $d\xi : TE \to T\mathbb{P}(E)$ of the vertical vector field $\widetilde{X}_F$ on E defined by $\widetilde{X}_F(v) = Fv$. Since the vertical sub-bundle $V_E \hookrightarrow TE$ may be identified canonically with the pullback bundle $\widetilde{\pi}^*E$, where $\widetilde{\pi} : E \to \Sigma$, $\widetilde{X}_F : E \to \widetilde{\pi}^*E$ is the composition of the map $F : E \to E$ with the canonical section $\sigma : E \to \widetilde{\pi}^*E$ given by $\sigma(v) = v$. Clearly, considered as a map $\mathcal{E} \to \widetilde{\pi}^*\mathcal{E}$, σ is holomorphic, so if $F : \mathcal{E} \to \mathcal{E}$ is holomorphic, the map $\widetilde{X}_F : \mathcal{E} \to \widetilde{\pi}^*\mathcal{E}$ is holomorphic. Then since $\xi : \mathcal{E} \to \mathbb{P}(\mathcal{E})$ is holomorphic, the map $T\mathcal{E} \to T^{1,0}(\mathbb{P}(\mathcal{E}))$ given by composing $d\xi$ with the isomorphism of smooth bundles $T\mathbb{P}(\mathcal{E}) \simeq T^{1,0}(\mathbb{P}(\mathcal{E}))$, is a holomorphic map, and since the image of $\widetilde{X}_F$ under this map is precisely $(X_F)^{1,0}$, we obtain the result. □

Lemma 4.8. *Suppose $g(\Sigma) \geq 2$. Assume that the Harder-Narasimhan filtration of the bundle $\mathcal{E}$ is equal to its Harder-Narasimhan-Seshadri filtration, so that in particular the slopes of the summands of $Gr(\mathcal{E})$ are all different. Then there are isomorphisms and equalities*

$$\begin{aligned}
H^0(T\mathbb{P}(\mathcal{E}_\infty)) &= \mathfrak{h}\left(\mathbb{P}(\mathcal{E}_\infty)\right) = \{(X_F)^{(1,0)} \mid F \in H^0(\mathcal{E}nd_0(\mathcal{E}))\} \simeq H^0(\mathcal{E}nd_0(\mathcal{E})),\\
\mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right) \oplus J_\infty \mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right) &= \{(X_F)^{(1,0)} \mid F \in \Gamma(End_0(E)),\ d_{A_\infty} F = 0\}\\
&= \{(X_F)^{(1,0)} \mid F \in \Gamma(End_0(E)),\ F = \oplus_i c_i Id_{E_i},\ c_i \in \mathbb{C}\} \simeq \mathbb{C}^m,\\
\mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right) &= \{\nabla^{1,0}_{g_{k,1}(J_\infty,h)} \Phi_h(iF) \mid F \in \Gamma(\mathfrak{u}(E)),\ d_{A_\infty} F = 0\},\\
&= \{(X_F)^{(1,0)} \mid F \in \Gamma(\mathfrak{u}(E)),\ d_{A_\infty} F = 0\}\\
&= \{(X_F)^{(1,0)} \mid F \in \Gamma(\mathfrak{u}(E)),\ F = \oplus_i c_i Id_{E_i},\ c_i \in i\mathbb{R}\} \simeq \mathbb{R}^m.
\end{aligned}$$

where m is the length of the Harder-Narasimhan filtration of $\mathcal{E}$. The space of Hamiltonian Killing fields on $\mathbb{P}(\mathcal{E}_\infty)$ is given by:

$$\begin{aligned}
&\mathfrak{ham}(J_\infty, g_{k,1}(J_\infty, h), \omega_{k,1}(J_\infty, h)))\\
= &\ \{J_\infty \nabla_{g_{k,1}(J_\infty,h)} \Phi_h(F) \mid F \in \Gamma(\mathfrak{u}(E)),\ d_{A_\infty} F = 0\}\\
= &\ \{J_\infty \nabla_{g_{k,1}(J_\infty,h)} \Phi_h(F) \mid F \in \Gamma(\mathfrak{u}(E)),\ F = \oplus_i c_i Id_{E_i}\ c_i \in i\mathbb{R}\}\\
= &\ \{X_F \mid F \in \Gamma(\mathfrak{u}(E)),\ F = \oplus_i c_i Id_{E_i}\} \simeq \mathbb{R}^m.
\end{aligned}$$

Therefore in particular we have

$$\ker \mathfrak{D}^*_{(\omega_{k,1}(J_\infty,h))} \mathfrak{D}_{(\omega_{k,1}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})} \simeq \mathbb{R}^{m+1}.$$

Proof. Exactly as in the proof of Lemma 4.3 we have an exact sequence

$$0 \longrightarrow H^0\left(\mathcal{V}_\infty\right) \longrightarrow H^0\left(T\mathbb{P}(\mathcal{E}_\infty)\right) \longrightarrow H^0\left(\pi^*\left(T\Sigma\right)\right) \longrightarrow \cdots,$$

and since $H^0\left(\pi^*\left(T\Sigma\right)\right) = 0$, and pushing forward the corresponding Euler sequence on $\mathbb{P}(\mathcal{E}_\infty)$ we obtain isomorphisms:

$$H^0\left(\mathcal{E}nd_0(\mathcal{E}_\infty)\right) \simeq H^0\left(\mathcal{V}_\infty\right) \simeq H^0\left(T\mathbb{P}(\mathcal{E}_\infty)\right),$$

with the map $H^0\left(\mathcal{E}nd(\mathcal{E}_\infty)\right) \to H^0\left(\mathcal{V}_\infty\right)$ being given by $F \mapsto (X_F)^{1,0}$, whose kernel may be identified with the constant multiples of the identity on $\mathcal{E}$, where by the previous lemma, this map is well-defined, and gives the above isomorphism. Then we have

$$H^0(T\mathbb{P}(\mathcal{E}_\infty)) = \{(X_F)^{1,0} \mid F \in H^0(\mathcal{E}nd(\mathcal{E}))\} = \mathfrak{h},$$

where the second equality comes from the fact that all vector fields of this form have zeros.

By the previous paragraph, we know that we may write any vector field in $H^0\left(T\mathbb{P}(\mathcal{E}_\infty)\right)$ as $(X_F)^{1,0}$ for $F \in H^0\left(\mathcal{E}nd(\mathcal{E}_\infty)\right)$. We will write $F = F_1 + F_2$, for $F_1 \in \Gamma(\mathfrak{u}(E))$ and $F_2 \in \Gamma(i\mathfrak{u}(E))$. Note that by Corollary 4.6

$$\begin{aligned}
X_{F_1} &= J_\infty\left(\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(-F_1)\right)_{\mathcal{V}_\infty}, X_{F_2} = \left(-i\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(F_2)\right)_{\mathcal{V}_\infty}\\
X_F &= J_\infty\left(\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(-F_1)\right)_{\mathcal{V}_\infty} - \left(i\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(F_2)\right)_{\mathcal{V}_\infty}
\end{aligned}$$

so that

$$(X_F)^{1,0} = \frac{1}{2}\left(J_\infty\left(\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(-F_1)\right)_{\mathcal{V}_\infty} - iJ_\infty\left(J_\infty\left(\nabla_{g_{k,1}(J_\infty,h)} \Phi_h(-F_1)\right)_{\mathcal{V}_\infty}\right)\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\left(-i\nabla_{g_{k,1}(J_\infty,h)}\Phi_h(F_2)\right)_{\mathcal{V}_\infty}-iJ_\infty\left(-i\nabla_{g_{k,1}(J_\infty,h)}\Phi_h(F_2)\right)_{\mathcal{V}_\infty}\right)\\
=\;&\frac{1}{2}\left(\nabla_{g_{k,1}(J_\infty,h)}i\Phi_h(-iF_1)\right)_{\mathcal{V}_\infty}-iJ_\infty\left(\nabla_{g_{k,1}(J_\infty,h)}i\Phi_h(-iF_1)\right)_{\mathcal{V}_\infty}\\
&+\frac{1}{2}\left(\left(\nabla_{g_{k,1}(J_\infty,h)}\Phi_h(-iF_2)\right)_{\mathcal{V}_\infty}-iJ_\infty\left(\nabla_{g_{k,1}(J_\infty,h)}\Phi_h(-iF_2)\right)_{\mathcal{V}_\infty}\right)\\
=\;&\left(\left(\nabla_{g_{k,1}(J_\infty,h)}(\Phi_h(-i(F_1+F_2)))\right)_{\mathcal{V}_\infty}\right)^{(1,0)}\\
=\;&\left(\left(\nabla_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF))\right)_{\mathcal{V}_\infty}\right)^{(1,0)}
\end{aligned}
$$

By the second part of Lemma 4.5 we obtain that if $d_{A_\infty}F=0$,

$$(d\Phi_h(-iF))_{\mathcal{H}_\infty}=\Phi_h\left(d_A\left(-iF\right)\right))=0.$$

Therefore

$$(\nabla_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF)))_{\mathcal{H}_\infty}=0,$$

and we obtain

$$
\begin{aligned}
(X_F)^{1,0} &= \nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF))\\
&= \nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF_1))+\nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF_2))\\
&= \nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF_1)+J_\infty\nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(F_2)),
\end{aligned}
$$

where we note that

$$\Phi_h(-iF_1),\Phi_h(F_2)$$

are imaginary valued. We therefore obtain

$$\{(X_F)^{(1,0)}\,|F\in\Gamma(End_0(E)),d_{A_\infty}F=0\}\subset\mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right)\oplus J_\infty\mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right).$$

Furthermore, if $F_2=0$, then

$$(X_F)^{1,0}=\nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF_1),$$

so if $d_{A_\infty}F=0$, then $(X_F)^{1,0}$ has imaginary holomorphy potential if and only if $F\in\Gamma(\mathfrak{u}(E))$, and we obtain

$$\{\nabla^{1,0}_{g_{k,1}(J_\infty,h)}\Phi_h(iF)\mid F\in\Gamma(\mathfrak{u}(E)),\ d_{A_\infty}F=0\}\subset\mathfrak{k}\left(\mathbb{P}(\mathcal{E}_\infty)\right)$$

On the other hand, suppose that

$$X_F=i\nabla^{1,0}_{g_{k,1}(J_\infty,h)}\left(\phi\right)$$

for some real valued function ϕ. Then since X_F is vertical, in particular we have

$$(\nabla^{1,0}_{g_{k,1}(J_\infty,h)}\left(\phi\right))_{\mathcal{H}_\infty}=0,$$

and by the above calculation

$$
\begin{aligned}
\left(\nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF))\right)_{\mathcal{V}_\infty} &= i\nabla^{1,0}_{g_{k,1}(J_\infty,h)}\left(\phi\right)\\
&\implies \left(\nabla_{g_{k,1}(J_\infty,h)}(\Phi_h(-iF))\right)_{\mathcal{V}_\infty}=i\nabla_{g_{k,1}(J_\infty,h)}\left(\phi\right)
\end{aligned}
$$

which means that

$$
\begin{aligned}
d\left(\Phi_h(-iF)\right)-i\phi &= \Phi_h(-id_{A_\infty}F)\\
&\implies \overline{\partial}_{J_\infty}\left(\Phi_h(-iF)\right)-i\phi)=\Phi_h(-i\partial_{A_\infty}F)=0,
\end{aligned}
$$

since F is holomorphic. Therefore $\Phi_h(-iF))$ and $i\phi$ differ by a constant and in particular

$$0 = d\,(i\phi))_{\mathcal{H}_\infty} = \Phi_h(-id_{A_\infty}F),$$

so we obtain $d_{A_\infty}F = 0$, and therefore we get the other inclusion

$$\mathfrak{k}\,(\mathbb{P}(\mathcal{E}_\infty)) \subset \{\nabla^{1,0}_{g_{k,1}(J_\infty,h)}\Phi_h(iF) \mid F \in \Gamma(\mathfrak{u}(E)),\ d_{A_\infty}F = 0\}.$$

The inclusion

$$\mathfrak{k}\,(\mathbb{P}(\mathcal{E}_\infty)) \oplus J_\infty\mathfrak{k}\,(\mathbb{P}(\mathcal{E}_\infty)) \subset \{(X_F)^{(1,0)}\,|F \in \Gamma(End_0(E)), d_{A_\infty}F = 0\}$$

follows in the same way. The equalities of these sets with

$$\begin{aligned} &\{(X_F)^{(1,0)} \mid F \in H^0(\mathcal{E}nd_0(\mathcal{E})),\ F = \oplus_i c_i Id_{E_i},\ c_i \in \mathbb{C}\},\\ \text{and } &\{(X_F)^{(1,0)} \mid F \in \Gamma(\mathfrak{u}(E)),\ F = \oplus_i c_i Id_{E_i},\ c_i \in i\mathbb{R}\} \end{aligned}$$

respectively, follow by Lemma 3.4

By Remark 2.3, using the description of $\mathfrak{k}$ above, there is a bijection

$$\begin{aligned} \mathfrak{k} &\to \mathfrak{ham}(J_\infty, g_{k,1}(J_\infty,h), \omega_{k,1}(J_\infty,h)))\\ \nabla^{1,0}_{g_{k,1}(J_\infty,h)}(\Phi_h(iF)) &\mapsto \frac{1}{2}J_\infty\nabla_{g_{k,1}(J_\infty,h)}\Phi_h(F). \end{aligned}$$

For $F \in \Gamma(\mathfrak{u}(E))$ and $d_{A_\infty}(F) = 0$, so we obtain the first description of $\mathfrak{ham}(J_\infty, g_{k,1}(J_\infty,h), \omega_{k,1}(J_\infty,h)))$. The second description of this space is obtained by applying Lemma 3.4. By the last part of Corollary 4.6, we obtain

$$X_F = J_\infty\left(\nabla_{g_{k,1}(J_\infty,h)}(\frac{1}{2}\Phi_h(F))\right),$$

so we obtain the third description as well. □

4.7. **Evolution equations.**

Lemma 4.9. *If h_t satisfies Hermitian-Yang-Mills flow then*

$$\frac{\partial}{\partial t}iF_{(h_t,\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} = 2i\bar{\partial}_J\partial_J\Phi_{h_t}(\Lambda_\omega F_{h_t}). \tag{4.14}$$

More generally, for any path of metrics h_t, we have that

$$\frac{\partial}{\partial t}iF_{(h_t,\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} = i\bar{\partial}_J\partial_J\Phi_{h_t}(ih_t^{-1}\partial_t h_t). \tag{4.15}$$

Proof. We have

$$\begin{aligned} 2\Phi_{h_t}((\Lambda_\omega F_{h_t} - i\mu(\mathcal{E})Id_E))([v]) &= i\frac{h_t(2(\Lambda_\omega F_{h_t} - i\mu(\mathcal{E})Id_E)v,v)}{\|v\|_{h_t}} = i\frac{h_t(ih_t^{-1}\partial_t h_t v,v)}{||v||_{h_t}}\\ = \quad -\frac{h_t(v,h_t^{-1}\frac{\partial h_t}{\partial t}v)}{\|v\|_{h_t}} &= -\frac{\partial h_t}{\partial t}(v,v)/\left\|v\right\|_{h_t}. \end{aligned}$$

If we define $f_t \in C^\infty(\mathbb{P}(E))$ by

$$f_t\,([v]) = log\left(\frac{h_t(v,v)}{h(v,v)}\right),$$

then the the relationship between the metrics $h_t^{\mathcal{L}}$ and $h^{\mathcal{L}}$ on the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \mathbb{P}(\mathcal{E})$ is given by $h_t^{\mathcal{L}} = e^{-f_t}h^{\mathcal{L}}$, and so we have

$$iF_{(h_t,\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} = iF_{(h,\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))} - i\bar{\partial}_J\partial_J f_t.$$

and clearly we also have

$$\frac{\partial f_t([v])}{\partial t} = \frac{\partial h_t}{\partial t}(v,v)/\left\|v\right\|_{h_t},$$

so that

$$2i\bar{\partial}_J\partial_J\Phi_{h_t}(\Lambda_\omega F_{h_t}) = 2i\bar{\partial}_J\partial_J\Phi_{h_t}(\Lambda_\omega F_{h_t} - i\mu(\mathcal{E})Id_E) = -\frac{\partial}{\partial t}i\bar{\partial}_J\partial_J f_t = \frac{\partial}{\partial t}iF_{(h_t,\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))}.$$

The more general statement follows from the exact same proof, as we have only used the Hermitan-Yang-Mills equations in the first line. ☐

Lemma 4.10. *Let g_t be the complex gauge transformations defined by equation 3.11, and $\widetilde{g}_t$ the induced diffeomorphisms. The (time-dependent) infinitesimal generator of the one parameter family of diffeomorphisms $\widetilde{g}_t$ is given by the vector field $-X_{i\Lambda_{\omega_\Sigma}F_{A_t}}$. That is, we have an equation*

$$\frac{\partial \widetilde{g}_t}{\partial t} = -X_{i\Lambda_{\omega_\Sigma}F_{A_t}}(\widetilde{g}_t). \tag{4.16}$$

In particular,

$$\frac{\partial\widehat{\omega}_{k,1}(t)}{\partial t} = \widetilde{g}_t^*\left(\frac{\partial\omega_{k,1}(t)}{\partial t} + \mathcal{L}_{-X_{i\Lambda_{\omega_\Sigma}F_{A_t}}}(\omega_{k,1}(t))\right),$$

so that

$$\begin{aligned}(\widetilde{g}_t^{-1})^*\left(\frac{\partial\widehat{\omega}_{k,1}(t)}{\partial t}\right) \overset{C^\infty}{\to} \mathcal{L}_{-X_{i\Lambda_{\omega_\Sigma}F_{A_\infty}}}(\omega_{k,1,\infty}) &= \mathcal{L}_{-\nabla_{g_{k,1}(J_\infty,h)}(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))}(\omega_{k,1,\infty}) \\ &= 2i\overline{\partial}_J\partial_J\left(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})\right).\end{aligned}$$

Proof. Let $F_t \in \Gamma(i\mathfrak{u}(E))$ be a one parameter family. Recall the vector fields $\widetilde{X}_{F_t} \in \Gamma(TE)$ defined by $v \mapsto F_t v$. Then with respect to the Riemannian metric on $TE \simeq \widetilde{\pi}^*E$ induced by h, these are the gradients of the functions

$$\begin{aligned}\widetilde{\Phi}_h(F_t) &: E \to \mathbb{R} \\ v &\mapsto h(F_t v, v),\end{aligned}$$

and the negative (time dependent) gradient flow of this path of functions is

$$\frac{\partial v_t}{\partial t} = -F_t v_t.$$

The projection of the gradient of $\widetilde{X}_{F_t}$ onto the unit sphere bundle $\mathbb{S}(E) \subset E$, is given by the vector field

$$w \mapsto \left(F_t - \frac{h(F_t w, w)}{h(w,w)}Id_E\right)w.$$

Because this vector field is homogenous, taking projections of both sides of the above flow to the sphere bundle, we see that the projection $w_t : \mathbb{R} \to \mathbb{S}(E)$ of the path v_t to $\mathbb{S}(E)$ solves the equation

$$\frac{\partial w_t}{\partial t} = -\left(F_t - \frac{h(F_t w_t, w_t)}{h(w_t,w_t)}Id_E\right)w_t.$$

In the same way, projecting to the projectivisation, the image $[w_t] : \mathbb{R} \to \mathbb{P}(E)$ satisfies the equation

$$\frac{\partial[w_t]}{\partial t} = -X_{F_t}([w_t]).$$

Now let g_t be the complex gauge transformations defining the Yang-Mills flow. By equation 3.11 we have that for any $v \in E$,

$$\frac{\partial g_t(v)}{\partial t} = -i\Lambda_{\omega_\Sigma}F_{A_t}(g_t(v)).$$

In other words $v_t = g_t(v)$ satisfies the (time-dependent) gradient flow equation above, and therefore we have

$$\frac{\partial [g_t(v)]}{\partial t} = -X_{i\Lambda_{\omega_\Sigma} F_{A_t}}\left([g_t(v)]\right)$$

for any v, and by the definition of $\widetilde{g}_t$ this says that

$$\frac{\partial \widetilde{g}_t}{\partial t}\left([v]\right) = -X_{i\Lambda_{\omega_\Sigma} F_{A_t}}\left(\widetilde{g}_t\left([v]\right)\right)$$

for every $[v] \in \mathbb{P}(E)$, which is precisely the stated result.

The second statement follows immediately from this, and formula for the time derivative of the pullback of family of differential forms by a family of diffemorphisms, the convergence of the Yang-Mills flow at infinity, and also the formula

$$\mathcal{L}_{\nabla_g \phi}\omega = 2i\partial_J\overline{\partial}_J\phi,$$

which is valid for any Kähler triple (g, ω, J) and any smooth function ϕ. □

4.8. **Decomposition of $C^\infty(\mathbb{P}(E))$.** Since the scalar curvature of the metrics we will construct is a smooth function on $\mathbb{P}(E)$, we will need a more precise description of the space of such functions. First we will consider the case when we fix the metric h and holomorphic structure $\bar{\partial}_A$ on E giving the holomorphic bundle $\mathcal{E}$, (inducing a holomorphic structure J on $P(E)$).

Note that we have a natural inclusion $\pi^* C^\infty(\Sigma) \hookrightarrow C^\infty(\mathbb{P}(E))$. We may define a map $\pi_{\Sigma *}$: $C^\infty(\mathbb{P}(E)) \to \pi^* C^\infty(\Sigma)$ by the pullback to $\mathbb{P}(E)$ of the integration over the fibres, namely

$$\pi_{\Sigma *}(f)([v]) = \pi^*\left(\int_{\mathbb{P}(E_z)} f \cdot \omega_{FS}^{r-1}\right),$$

where $\pi([v]) = z$.

Clearly for $f \in \pi^* C^\infty(\Sigma)$, we have $f = \frac{1}{vol(\mathbb{P}^{r-1})}\pi_{\Sigma *}(f)$. If we denote by $C_0^\infty(\mathbb{P}(E)) \hookrightarrow C^\infty(\mathbb{P}(E))$ the subspace of smooth functions whose restriction to each fibre has mean value zero, and define $p : C^\infty(\mathbb{P}(E)) \to C_0^\infty(\mathbb{P}(E))$, by $p(f) = f - \frac{1}{vol(\mathbb{P}^{r-1})}\pi_{\Sigma *}(f) \in C_0(\mathbb{P}(E))$, there is an exact sequence

$$0 \to \pi^* C^\infty(\Sigma) \to C^\infty(\mathbb{P}(E)) \xrightarrow{p} C_0^\infty(\mathbb{P}(E)) \to 0,$$

and the inclusion $C_0^\infty(\mathbb{P}(E)) \hookrightarrow C^\infty(\mathbb{P}(E))$ gives a splitting:

$$C^\infty(\mathbb{P}(E)) = \pi^* C^\infty(\Sigma) \oplus C_0^\infty(\mathbb{P}(E)),$$

corresponding to the fact that each function f can be written as

$$f = \frac{1}{vol(\mathbb{P}^{r-1})}\pi_{\Sigma *}(f) + p(f).$$

There is a further decomposition of $C_0^\infty(\mathbb{P}(E))$ as follows. Denote by $\Phi_h\left(\Gamma(\mathfrak{su}(E,h)\right)$, the set of C^∞ functions in the image of the traceless endomorphisms of E under Φ_h. For each fibre $\mathbb{P}(E_z)$ one can calculate

$$\int_{\mathbb{P}(E_z)} \Phi_h(F) \cdot \omega_{FS}^{r-1} = 0,$$

so that there is an inclusion $\Phi_h\left(\Gamma(\mathfrak{su}(E,h)\right) \hookrightarrow C_0^\infty(\mathbb{P}(E))$. Then we have a splitting

$$C^\infty(\mathbb{P}(E)) = \pi^* C^\infty(\Sigma) \oplus \Phi_h\left(\Gamma(\mathfrak{su}(E,h)\right) \oplus C_h^\infty(\mathbb{P}(E))_\perp,$$

where $C_h^\infty(\mathbb{P}(\mathcal{E}))_\perp$ is the set of functions which are fibrewise L^2 orthogonal to $\pi^* C^\infty(\Sigma) \oplus \Phi_h\left(\Gamma(\mathfrak{u}(E,h)\right)$. Note that $\Phi_h\left(\Gamma(\mathfrak{u}(E,h)\right)$ and $C_h^\infty(\mathbb{P}(\mathcal{E}))_\perp$ depend on the Hermitian metric h.

The functions $\pi^* C^\infty(\Sigma)$ are constant on the fibres. The following lemma says in particular that the space $\Phi_h(\mathfrak{su}(E))$ is also finite dimensional when restricted to the fibres. For the proof see for example the thesis of Pook.

Lemma 4.11. *For any $z \in \Sigma$, the space*

$$\Phi_h(\mathfrak{su}(E_z)) := \Phi_h(\mathfrak{su}(E))|_{\mathbb{P}(E_z)} := \{\Phi_h(F)|_{\mathbb{P}(E_z)} | F \in \mathfrak{su}(E)\}$$

is exactly the lowest eigenspace (corresponding to the eigenvalue r) of the Laplacian with respect to the Fubini-Study metric on $\mathbb{P}(E_z)$.

The decomposition of the $C^\infty(\mathbb{P}(E))$ behaves well with respect to the gauge transformations g_t and the induced diffeomorphisms $\widetilde{g}_t$. Namely, we have the following lemma which is a simple consequence of the definitions.

Lemma 4.12. *For any $g \in \mathcal{G}^{\mathbb{C}}(E)$, with $\widetilde{g} \in \mathcal{G}^{\mathbb{C}}(\mathbb{P}E)$ its induced diffeomorphism, and any endomorphism $F \in \mathfrak{su}(E, h)$, we have*

$$\widetilde{g}^* \left(\Phi_h(F)\right) = \Phi_{g \cdot h}\left(g^*(F)\right),$$

In particular, if $g_t \in \mathcal{G}^{\mathbb{C}}(E)$ is the path of gauge transformations associated to equation 3.10, then:

$$\widetilde{g}_t^* \left(\Phi_h\left(\Lambda_{\omega_\Sigma} F_{A_t}\right)\right) = \Phi_{h_t}\left(\Lambda_{\omega_\Sigma} F_{h_t}\right). \tag{4.17}$$

As a result, for each t we have a splitting

$$\begin{aligned} C^\infty(\mathbb{P}(E)) &= \pi^* C^\infty(\Sigma) \oplus \tilde{g}_t^* \left(\Phi_h\left(\Gamma(\mathfrak{su}(E,h))\right)\right) \oplus \tilde{g}_t^* \left(C_h^\infty(\mathbb{P}(E))_\perp\right) \\ &= \pi^* C^\infty(\Sigma) \oplus \Phi_{h_t}\left(\mathfrak{su}(E, h_t)\right) \oplus C_{h_t}^\infty(\mathbb{P}(E))_\perp. \end{aligned}$$

We may therefore write any $\Psi \in C^\infty(\mathbb{P}(E))$ as

$$\begin{aligned} \Psi &= \Psi_\Sigma + \tilde{g}_t^*\left(\Psi_{\Phi_h}\right) + \tilde{g}_t^*\left(\Psi_\perp\right) \\ &= \Psi_\Sigma + \hat{\Psi}_{\Phi_h} + \hat{\Psi}_\perp = \hat{\Psi}. \end{aligned}$$

Finally, we will need the following lemma.

Lemma 4.13. *The projection maps $\pi_{\Sigma *} : C^\infty(\mathbb{P}(E)) \to \pi^* C^\infty(\Sigma)$, $\pi_{\Phi_h *} : C^\infty(\mathbb{P}(E)) \to \Phi_h(\mathfrak{su}(E))$, $\pi_{\perp *} : C^\infty(\mathbb{P}(E)) \to C^\infty(\mathbb{P}(E))_\perp$ onto the three components in this decomposition are continuous with respect to the frechét topologies. In particular, if*

$$\Psi(t) = \Psi_\Sigma(t) + \Psi_{\Phi_h}(t) + \Psi_\perp(t)$$

and $\Psi(t) \to \Psi_\infty$ in $C^\infty(\mathbb{P}(E))$, then $\Psi_\Sigma(t) \to \Psi_{\Sigma,\infty}$, $\Psi_{\Phi_h}(t) \to \Psi_{\Phi_h,\infty}$, and $\Psi_\perp(t) \to \Psi_{\perp,\infty}$, where $\Psi_{\Sigma,\infty}, \Psi_{\Phi_h,\infty}$, and $\Psi_{\perp,\infty}$ are the images under the respective projections of Ψ_∞. Moreover, the rate of convergence is preserved under the projections.

Proof. We may define an infinite rank vector bundle $W \to \Sigma$, with fibres $W_z := C^\infty(\mathbb{P}(E_z))$, so that the space smooth sections $C^\infty(W)$ may be identified with $C^\infty(\mathbb{P}(E))$ by the isomorphism $\tau : C^\infty(W) \to C^\infty(\mathbb{P}(E))$ defined by $\tau(\sigma)([v]) = \sigma(z)([v])$, where $\pi([v]) = z$. There is a decomposition

$$W = \underline{\mathbb{C}} \oplus W_{\Phi_h} \oplus W_\perp$$

corresponding to the fibrewise decomposition $W_z = \mathbb{C} \oplus \Phi_h(\mathfrak{su}(E_z)) \oplus C^\infty(\mathbb{P}(E_z))_\perp$. The map τ identifies $C^\infty(\underline{\mathbb{C}}) = C^\infty(\Sigma)$,$C^\infty(W_{\Phi_h})$, and $C^\infty(W_\perp)$ with $\pi^* C^\infty(\Sigma)$, $\Phi_h(\mathfrak{su}(E))$, and $C^\infty(\mathbb{P}(E))_\perp$ respectively. If we write π_Σ, π_{Φ_h}, $\pi_\perp$ for the three projection maps from W onto $\underline{\mathbb{C}}$, W_{Φ_h}, and $W_\perp$, then the projections $\pi_{\Sigma *}$, $\pi_{\Phi_h *}$, and $\pi_{\perp *}$ are the compositions with τ of the maps $C^\infty(W) \to C^\infty(\Sigma)$, $C^\infty(W) \to C^\infty(W_{\Phi_h})$, and $C^\infty(W) \to C^\infty(W_\perp)$ induced by the projections. Since these latter maps are the induced map of smooth bundle morphisms, they are continuous with respect to the frechét topologies. The other statements follow automatically. □

4.9. **Continuity of Φ_h.** In this subsection we show that the map $\Phi_h : \mathfrak{su}(E,h) \to C^\infty(\mathbb{P}(E))$ behaves well with respect to convergence in the C^∞ topologies. We will use the metric $g_{k,1,\infty}$ (which depends on k) on $C^\infty(\mathbb{P}(E))$ to compute covariant derivatives, so we will need to estimate C^k norms with respect to this metric uniformly in terms of the norm of a fixed metric independent of k. For this we will give a slight modification of a result in [F].

By analogy with the discussion in Section 5.1 of [F], in the following we will consider a ball $B \subset \Sigma$, centred at x_0, such that there is a biholomorphism $\mathbb{P}(\mathcal{E}_\infty)|_B = B \times \mathbb{P}^{r-1}$, where $\mathcal{E}_\infty = (E, \overline{\partial}_{A_\infty})$ corresponding to the limit A_∞ of the Yang-Mills flow. This biholomorphism will be arranged so that the horizontal distribution on the central fibre $\mathbb{P}^{r-1}_{x_0}$ is equal to the restriction of the restriction of the second factor in the decomposition

$$T(\mathbb{P}(\mathcal{E}_\infty)|_B) \cong T\mathbb{P}^{r-1} \oplus TB.$$

We will compare the restriction of the Kähler form $\omega_{k,1,\infty}$ to $\mathbb{P}(\mathcal{E}_\infty)|_B$ with the product $\omega_k' = \omega_{FS} \oplus k\omega_B$, compatible with the split complex structure $J_{\mathbb{P}^{r-1}} \oplus J_B$, where ω_{FS} is the usual Fubini-Study form on the fibre, and ω_B is the flat Kähler form on B agreeing with ω_Σ at the origin.

Lemma 4.14. *Let $V \to \Sigma$ be a smooth vector bundle and consider the pullback $\pi^*(V) \to \mathbb{P}(E)$, and $\beta \in C^k((T^*\mathbb{P}(E))^{\otimes i} \otimes \pi^*(V))$ with $\beta = \pi^*(\alpha)$ for $\alpha \in C^k((T^*\Sigma)^{\otimes i} \otimes V)$. Then we have an estimate of the form*

$$\|\beta\|_{C^s(g_{k,1,\infty})} \leq Ck^{-i/2} \|\alpha\|_{C^s(g_\Sigma)}.$$

If β is not a pullback then we still have

$$\|\beta\|_{C^s(g_{k,1,\infty})} = \mathcal{O}(1).$$

Proof. First we remark that the result is true for the product metric $g_k^{'} = g_{FS} \oplus kg_B$ on $B \times \mathbb{P}^{r-1}$. This is because for the Levi-Civita connection $\nabla_{g_k^{'}}$ for this metric (coupled to the pullback of any connection on V) is the direct sum $\nabla_{g_{FS}} \oplus k\nabla_{g_B}$ of the two Levi-Civita connections on each factor (coupled to the pullback connection on V), with the second factor weighted by k. Since β is pulled back from the base, it is constant on the fibres we have that $\nabla_{g_k^{'}}(\beta) = k\pi^*(\nabla_{g_B}\alpha)$, and similarly $\nabla^s_{g_k^{'}}(\beta) = k^s\pi^*(\nabla_{g_B}\alpha)$, and therefore since the expression for the pointwise norm $|-|_{g_k^{'}}$ of an $s+i$ tensor involves the inverse of the metric $2(s+i)$ times, we obtain

$$\|\beta\|_{C^s(g_k')} \leq \left(k^s k^{-(s+i)}\right)^{1/2} \|\beta\|_{C^s(g_B)} = k^{-i/2} \|\alpha\|_{C^s(g_B)}.$$

Moreover, if the ball B is taken to be small enough, the norms $\|-\|_{C^s(g_B)}$ and $\|-\|_{C^s(g_\Sigma)}$ are uniformly equivalent, and therefore we also have

$$\|\beta\|_{C^s(g_k')} \leq k^{-i/2} \|\alpha\|_{C^s(g_\Sigma)}.$$

By the slightly more sophisticated argument of [F] Theorem 5.2, we in fact have that $\|-\|_{C^s(g_k')}$ and $\|-\|_{C^s(g_{k,1,\infty})}$ are uniformly equivalent as well, so on a very small ball B we obtain

$$\|\beta\|_{C^s(g_{k,1,\infty})} \leq k^{-i/2} \|\alpha\|_{C^s(g_\Sigma)}.$$

Covering $\mathbb{P}(E)$ by charts of this kind, we obtain the global estimate of this form.

The proof for the case when β is not a pullback is exactly the same except that there are terms involving the $\nabla_{g_{FS}}$ as well, but these do not depend on k. □

Lemma 4.15. *Let $F_t \in \mathfrak{su}(E,h)$ be a path of endomorphisms, and F_∞ a fixed endomorphism. Then $F_t \to F_\infty$ in the C^∞ topology with respect to the metric g_Σ, at a rate of $f(t)$, that is, for each $s \geq 0$ and for $t >> 0$:*

$$\|F_t - F_\infty\|_{C^s(\mathfrak{su}(E,h),g_\Sigma)} \leq Cf(t),$$

if and only if $\Phi_h(F_t)$ converges to $\Phi_h(F_\infty)$ in the C^∞ with respect to the metric $g_{k,1,\infty}$ at the same rate, that is; for each $s \geq 0$ and for $t >> 0$:

$$\|\Phi_h(F_t) - \Phi_h(F_\infty)\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} \leq Cf(t).$$

In particular for the path of endomorphisms given by $\Lambda_{\omega_\Sigma} F_{A_t}$ where A_t is given by the Yang-Mills flow, we have for each $s \geq 0$ and for $t >> 0$:

$$\|\Phi_h(\Lambda_{\omega_\Sigma} F_{A_t}) - \Phi_h(\Lambda_{\omega_\Sigma} F_{A_\infty})\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} \leq C/\sqrt{t}$$

Proof. We consider the pullback bundle $\pi^*(\mathfrak{su}(E,h)) \to \mathbb{P}(E)$ via the map $\pi : \mathbb{P}(E) \to \Sigma$. By construction, a point in $\pi^*(\mathfrak{su}(E,h))$ is a pair $([v], F) \in \mathbb{P}(E) \times \pi^*(\mathfrak{su}(E,h))$, and therefore Φ_h induces a bundle map

$$\Xi : \pi^*(\mathfrak{su}(E,h)) \to \underline{\mathbb{C}}$$

defined by $\Xi([v], F) = \Phi_h(F)([v])$. This is obviously linear on the fibres, and smooth by the definition of $\Phi_h(F)([v])$, and in turn induces a linear map on the spaces of C^∞ sections

$$\Xi_* : C^\infty(\pi^*(\mathfrak{su}(E,h))) \to C^\infty(\underline{\mathbb{C}}) = C^\infty(\mathbb{P}(E))$$

given by $\Xi_*(\sigma)([v]) = \Xi(\sigma(v))$. Given any $F \in \mathfrak{su}(E,h)$ we may define a smooth section σ_F of $\pi^*(\mathfrak{su}(E,h))$ by $\sigma_F([v]) := ([v], F)$ (which is exactly the section $\pi^*(F)$), and therefore we have

$$\Xi_*(\sigma_F)([v]) = \Xi([v], F) = \Phi_h(F)([v])$$

for all $[v] \in \mathbb{P}(E)$, and so $\Phi_h(F) = \Xi_*(\sigma_F)$. Note that Ξ_* is bounded (and therefore continuous) with respect to the Banach space topologies on $C^s(\pi^*(\mathfrak{su}(E,h)))$ and $C^s(\mathbb{P}(E))$ for each s and is therefore continuous with respect to the Fréchet topologies on $C^\infty(\pi^*(\mathfrak{su}(E,h)))$ and $C^\infty(\mathbb{P}(E))$ induced by the semi-norms defined by

$$\left\|\nabla^s_{g_\Sigma}\sigma\right\|_{C^0(\pi^*(\mathfrak{su}(E,h)),g_{k,1,\infty})} \text{ and } \left\|\nabla^s_{g_{k,1,\infty}}\gamma\right\|_{C^0(\mathbb{P}(E),g_{k,1,\infty})}$$

as s ranges over all positive integers. This follows since for each s we have

$$\|\Xi(\sigma)\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} \leq C\,\|\Xi\|_{C^s}\,\|\sigma\|_{C^s(\pi^*(\mathfrak{su}(E,h),g_{k,1,\infty})} \leq C\,\|\sigma\|_{C^s(\pi^*(\mathfrak{su}(E,h),g_{k,1,\infty})}.$$

Therefore if $F_t \to F_\infty$ smoothly then $\sigma_{F_t} \to \sigma_{F_\infty}$ smoothly and so $\Phi_h(F_t) \to \Phi_h(F_\infty)$ smoothly as well.

Since Ξ_* is linear we have for each s

$$\begin{aligned}
\|\Phi_h(F_t) - \Phi_h(F_\infty)\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} &= \|\Xi_*(\sigma_{F_t}) - \Xi_*(\sigma_{F_\infty})\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} \\
&= \|\Xi_*(\sigma_{F_t} - \sigma_{F_\infty})\|_{C^s(\mathbb{P}(E),g_{k,1,\infty})} \\
&\leq C\,\|\sigma_{F_t - F_\infty}\|_{C^s(\pi^*(\mathfrak{su}(E,h),g_{k,1,\infty})} \\
&\leq C\,\|F_t - F_\infty\|_{C^s(\mathfrak{su}(E,h),g_\Sigma)} \leq Cf(t),
\end{aligned}$$

for t sufficiently large.

To prove the converse, we note that Ξ_* is invertible since

$$\Xi_*(\sigma_F)([v]) = \Phi_h(F)([v]) = \sqrt{-1}\frac{h_s(Fv, v)}{h_s(v,v)} = 0$$

for all $[v]$ if and only if $F = 0$ if and only if σ_F, so Ξ_* is injective; and given $f \in C^\infty(\mathbb{P}(E))$,

$$\Xi_*(f\sigma_{Id_E})([v]) = \sqrt{-1}\frac{h_s(f([v])v, v)}{h_s(v,v)} = f([v]),$$

so $\Xi_*(f\sigma_{Id_E}) = f$, and Ξ_* is surjective. Therefore by the bounded inverse theorem the inverse maps

$$\Xi_*^{-1} : C^s(\underline{\mathbb{C}}) = C^s(\mathbb{P}(E)) \to C^s(\pi^*(\mathfrak{su}(E,h)))$$

are bounded (and therefore continuous) for each s, and so the inverse is continous with respect to the Fréchet topologies on $C^\infty(\pi^*(\mathfrak{su}(E,h)))$ and $C^\infty(\mathbb{P}(E))$. Therefore if $\Phi_h(F_t) \to \Phi_h(F_\infty)$ smoothly, then $F_t \to F_\infty$ smoothly. By the previous argument an estimate

$$\|\Phi_h(F_t) - \Phi_h(F_\infty)\|_{C^s(\mathbb{P}(E), g_{k,1,\infty})} \leq Cf(t)$$

for all s implies

$$\|F_t - F_\infty\|_{C^s(\mathfrak{su}(E,h), g_\Sigma)} \leq Cf(t)$$

for all s.

The last statement follows from Theorem 3.5. □

4.10. **Convergence of various quantities and operators.**

Lemma 4.16. *If J_t is the holomorphic structure on $\mathbb{P}(\mathcal{E}_t)$, where $\mathcal{E}_t = (E, \overline{\partial}_{A_t})$ with A_t satisfying the Yang-Mills flow, we have for every j and m, for $t >> 0$:*

$$\left\|\partial_t^j \left(J_t - J_\infty\right)\right\|_{C^m(g_{k,1\infty})}, \left\|\partial_t^j \left(\omega_{k,1}(t) - \omega_{k,1,\infty}\right)\right\|_{C^m(g_{k,1\infty})} \leq C/\sqrt{t},$$

where the constant C is independent of k, and in particular, for all p, q, and ε

$$\|J_t - J_\infty\|_{W_{4,p+1,q,w_\varepsilon(t)}(g_\infty)}, \|\omega_{k,1}(t) - \omega_{k,1,\infty}\|_{W_{4,p+1,q,w_\varepsilon(t)}(g_\infty)} < \infty.$$

Consequently, for any smooth function ϕ, we have we also have

$$\begin{aligned} \left\|\partial_t^j \left(\overline{\partial}_{J_t} - \overline{\partial}_{J_\infty}\right)\phi\right\|_{C^m(g_{k,1,\infty})} &\leq 1/\sqrt{t} \\ \text{and } \left\|\left(\overline{\partial}_{J_t} - \overline{\partial}_{J_\infty}\right)\phi\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} &< \infty. \end{aligned}$$

Proof. We have by definition

$$\begin{aligned} &\omega_{k,1}(t) - \omega_{k,1,\infty} \\ = \;& \omega_{FS}(h) + (\Phi_h(\Lambda_{\omega_\Sigma} F_{A_t}) + k)\omega_\Sigma - (\Phi_h(\Lambda_{\omega_\Sigma} F_{A_\infty}) + k)\omega_\Sigma - \omega_{FS}(h) \\ = \;& (\Phi_h(\Lambda_{\omega_\Sigma} F_{A_t}) - \Phi_h(\Lambda_{\omega_\Sigma} F_{A_\infty}))\,\omega_\Sigma. \end{aligned}$$

From Lemma 4.15 we obtain the second stated inequality.

For the first inequality we recall that by definition J_t and J_∞ are defined using the smooth splittings

$$T\mathbb{P}(E) = V \oplus H_t = V \oplus H_\infty,$$

where H_t and H_∞ are complementary subbundles to V determined by A_t and A_∞ respectively. Namely, for any smooth vector field $X \in \Gamma(T\mathbb{P}(E))$, we may write

$$X = X_V + X_{H_t} = X_V + X_{H_\infty}$$

according to these splittings, and then

$$\begin{aligned} J_t(X) &= j_V(X_V) + J_\Sigma(X_{H_t}) \\ &= j_V(X_V) + J_\Sigma(X_{H_\infty}), \end{aligned}$$

where j_V is the complex structure on V induced from the complex structure on $V_E = \widetilde{\pi}^*(E)$ determined by multiplication by i on the fibres. Then to compare J_t and J_∞ it suffices to compare X_{H_t} and X_{H_∞}. These vector fields correspond to vector fields $\widetilde{X}_{H_t} \in \Gamma\left(H_E^{A_t}\right)$and $\widetilde{X}_{H_\infty} \in \Gamma\left(H_E^{A_\infty}\right)$ under the smooth isomorphisms $H_E^{A_t} \simeq H_t$ and $H_E^{A_\infty} \simeq H_\infty$, which are horizontal lifts of some vector field $X_\Sigma \in \Gamma\left(T\Sigma\right)$. Now note that for any smooth section $\sigma \in \Gamma(E)$, we may define a vertical vector field $\widehat{\sigma} \in \Gamma\left(V_E\right)$, by $\widehat{\sigma}(v) = \sigma(x)$, where $v \in E_x$. Then by construction we have that

$$\nabla_{A_t}\widehat{(\sigma)\left(X_\Sigma\right)} = [\widetilde{X}_{H_t}, \widehat{\sigma}], \nabla_{A_\infty}\widehat{(\sigma)\left(X_\Sigma\right)} = [\widetilde{X}_{H_\infty}, \widehat{\sigma}],$$

so that

$$\nabla_{A_t} - \widehat{\nabla_{A_\infty}(\sigma)}\left(X_\Sigma\right) = [\widetilde{X}_{H_t} - \widetilde{X}_{H_\infty}, \widehat{\sigma}].$$

In other words we can think of both $\left(\nabla_{A_t} - \nabla_{A_\infty}\right) X_\Sigma$ and $\widetilde{X}_{H_t} - \widetilde{X}_{H_\infty}$ as being maps $\Gamma(E) \to \Gamma\left(V_E\right)$, and these maps are equal. From this we obtain an inequality:

$$\begin{aligned}
\left\|\partial_t^j\left(J_t - J_\infty\right) X\right\|_{C^m(g_{k,1,\infty})} &\leq C\left\|J_\Sigma\left(\partial_t^j\left(X_{H_t} - X_{H_\infty}\right)\right)\right\|_{C^m(g_\Sigma)} \leq C\left\|\partial_t^j\left(\widetilde{X}_{H_t} - \widetilde{X}_{H_\infty}\right)\right\|_{C^m(g_\Sigma)} \\
&\leq C\left(\left\|\partial_t^j\left(A_t - A_\infty\right)\right\|_{C^m(g_\Sigma)}\right),
\end{aligned}$$

and the result now follows from Lemma 3.6. The final statement of the lemma follows directly from this and Equation 4.3. $\square$

We define the operators $\Delta_{\mathcal{H}_t}$ and $\Delta_{\mathcal{V}_t}$ by

$$\begin{aligned}
\Delta_{\mathcal{H}_t}\left(\phi\right) &= i\Lambda_{\omega_\Sigma}\left(i\overline{\partial}_{J_t}\partial_{J_t}\phi\right)_{\mathcal{H}_t\mathcal{H}_t} \\
\Delta_{\mathcal{V}_t}\left(\phi\right) &= i\Lambda_{\omega_{FS(h,J_t)}}\left(i\overline{\partial}_{J_t}\partial_{J_t}\phi\right)_{\mathcal{V}_t\mathcal{V}_t},
\end{aligned}$$

and $\Delta_{\mathcal{H}_\infty}$ and $\Delta_{\mathcal{V}_\infty}$ in the same way.

Lemma 4.17. *For every j and m we have and every $\phi \in C^m(g_{k,1\infty})$, for $t >> 0$:*

$$\begin{aligned}
\left\|\partial_t^j\left(\Delta_{\omega_{k,1}(t)} - \Delta_{\omega_{k,1,\infty}}\right)(\phi)\right\|_{C^m(g_{k,1\infty})} &\leq C\left\|\phi\right\|_{C^m(g_{k,1\infty})}/\sqrt{t}, \\
\left\|\partial_t^j\left(\Delta_{\mathcal{V}_t} - \Delta_{\mathcal{V}_\infty}\right)(\phi)\right\|_{C^m(g_{k,1\infty})} &\leq C\left\|\phi\right\|_{C^m(g_{k,1\infty})}/\sqrt{t}, \\
\left\|\partial_t^j\left(\Delta_{\mathcal{H}_t} - \Delta_{\mathcal{H}_\infty}\right)(\phi)\right\|_{C^m(g_{k,1\infty})} &\leq C\left\|\phi\right\|_{C^m(g_{k,1\infty})}/\sqrt{t},
\end{aligned}$$

where the constant C is independent of k.

Proof. We remark first of all that for any Kähler metric g with Kähler form ω, the volume form ω^r is parallel, and therefore for any covariant derivative ∇_g^i, and any function f we have

$$\nabla_g^i\left(f \cdot \omega^r\right) = \nabla_g^i\left(f\right) \otimes \omega^r,$$

so that

$$\begin{aligned}
\left\|f \cdot \omega^r\right\|_{C^m(g)} &= \left(\left\|f\right\|_{C^0(g)} + \cdots + \left\|\nabla_g^m f\right\|_{C^o(g)}\right)\left\|\omega^r\right\|_{C^0(g)} \\
&= \left\|f\right\|_{C^m(g)},
\end{aligned}$$

since $\left\|\omega^r\right\|_{C^0(g)} = 1$. In particular, for a top degree form β we have

$$\left\|\beta\right\|_{C^m(g)} = \left\|\frac{\beta}{\omega^r} \cdot \omega^r\right\|_{C^m(g)} = \left\|\frac{\beta}{\omega^r}\right\|_{C^m(g)}$$

Then for any j and m we have:

$$
\begin{aligned}
&\left\| \partial_t^j \left(\left(\Delta_{\omega_{k,1}(t)} - \Delta_{\omega_{k,1,\infty}} \right) (\phi) \right) \right\|_{C^m(g_{k,1\infty})} \\
= \; & C \left\| \partial_t^j \left(\frac{\omega_{k,1}(t)^{r-1} \wedge i\bar{\partial}_{J_t}\partial_{J_t}(\phi)}{(\omega_{k,1}(t))^r} - \frac{(\omega_{k,1,\infty})^{r-1} \wedge i\bar{\partial}_{J_\infty}\partial_{J_\infty}(\phi)}{\left(\omega''_{k,1,\infty}\right)^r} \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
= \; & C \left\| \partial_t^j \left(\frac{\frac{\left(\omega_{k,1,\infty}\right)^r}{\left(\omega_{k,1}(t)\right)^r}\omega_{k,1}(t)^{r-1} \wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi) - (\omega_{k,1,\infty})^{r-1} \wedge i\bar{\partial}_{J_\infty}\partial_{J_\infty}(\phi)}{(\omega_{k,1,\infty})^r} \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
= \; & C \left\| \partial_t^j \left(\frac{(\omega_{k,1,\infty})^r}{(\omega_{k,1}(t))^r}\omega_{k,1}(t)^{r-1} \wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi) - (\omega_{k,1,\infty})^{r-1} \wedge i\bar{\partial}_{J_\infty}\partial_{J_\infty}(\phi) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
\leq \; & C \left\| \partial_t^j \left(\left(\frac{(\omega_{k,1,\infty})^r}{(\omega_{k,1}(t))^r}\omega_{k,1}(t)^{r-1} - (\omega_{k,1,\infty})^{r-1} \right) \wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
& + C \left\| \partial_t^j \left((\omega_{k,1,\infty})^{r-1} \wedge \left(i\overline{\partial}_{J_t}\partial_{J_t}(\phi) - i\bar{\partial}_{J_\infty}\partial_{J_\infty} \right)(\phi) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
\leq \; & C \left\| \partial_t^j \left(\frac{(\omega_{k,1,\infty})^r}{(\omega_{k,1}(t))^r} \left(\omega_{k,1}(t)^{r-1} - (\omega_{k,1,\infty})^{r-1} \right) \wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
& + C \left\| \partial_t^j \left(\left((\omega_{k,1,\infty})^r - (\omega_{k,1}(t))^r \right) \frac{((\omega_{k,1,\infty})^{r-1}) \wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi)}{(\omega_{k,1}(t))^r}) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
& + C \left\| \partial_t^j \left((\omega_{k,1,\infty})^{r-1} \wedge \left(i\overline{\partial}_{J_t}\partial_{J_t}(\phi) - i\bar{\partial}_{J_\infty}\partial_{J_\infty} \right)(\phi) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} \\
\leq \; & C \left(\sum_{i=0}^{j} \left\| \partial_t^i \left(\omega_{k,1}(t) - (\omega_{k,1,\infty}) \right) \right\|_{C^m\left((g_{k,1,\infty})\right)} + \left\| \partial_t^i a_t^{0,1} \right\|_{C^m\left((g_{k,1,\infty})\right)} \right) \|\phi\|_{C^m(g_{k,1\infty})} \\
& + C \left(\sum_{i=0}^{j} \left\| \partial_t^i a_t^{1,0} \right\|_{C^m\left((g_{k,1,\infty})\right)} \|\phi\|_{C^m(g_{k,1\infty})} + \left\| \partial_t^i a_t^{1,0} \right\|_{C^m\left((g_{k,1,\infty})\right)} \cdot \left\| \partial_t^i a_t^{0,1} \right\|_{C^m\left((g_{k,1,\infty})\right)} \|\phi\|_{C^m(g_{k,1\infty})} \right) \\
\leq \; & C \|\phi\|_{C^m(g_{k,1\infty})} / \sqrt{t},
\end{aligned}
$$

where as usual $\overline{\partial}_{J_t} = \overline{\partial}_{J_\infty} + a_t^{0,1}$, $\partial_{J_t} = \partial_{J_\infty} + a_t^{1,0}$ so that $a_t^{0,1}$ and $a_t^{1,0}$ and all of their time derivatives converge to 0 smoothly at a rate of $\frac{1}{\sqrt{t}}$, and where we have used the convergence of all the quantities that appear in the above formulas as $s \to \infty$, as well as the previous lemma.

A simple calculation shows (see equation 5.20 below and substitute in $i\overline{\partial}_{J_t}\partial_{J_t}(\phi)$ for the precise formula) that for k sufficiently large

$$\Delta_{\omega_{k,1}(t)} = \Delta_{\mathcal{V}_t}(\phi) + k^{-1}\Delta_{\mathcal{H}_t}(\phi) + \mathcal{O}(k^{-2}),$$

and the same formula holds for $\Delta_{\omega_{k,1,\infty}}$. Then we have

$$\left(\Delta_{\omega_{k,1}(t)} - \Delta_{\omega_{k,1,\infty}} \right)(\phi) = \left(\Delta_{\mathcal{V}_t} - \Delta_{\mathcal{V}_\infty} \right)(\phi) + k^{-1}\left(\Delta_{\mathcal{H}_t} - \Delta_{\mathcal{H}_\infty} \right)(\phi) + \mathcal{O}(k^{-2}).$$

In particular the constant in the above inequality is independent of k. Moreover, taking $k \to \infty$ we get the second inequality in the statement of the lemma. Finally, we may write

$$\left(\Delta_{\mathcal{H}_t} - \Delta_{\mathcal{H}_\infty} \right)(\phi) + \mathcal{O}(k^{-1})$$

$$= \quad k\left(\Delta_{\omega_{k,1}(t)} - \Delta_{\omega_{k,1,\infty}} + \Delta_{\mathcal{V}_\infty} - \Delta_{\mathcal{V}_t}\right)(\phi),$$

where the right hand side is $\mathcal{O}(1)$ and satisfies the required estimate. Then again taking $k \to \infty$ we get the last inequality in the statement of the lemma. □

Lemma 4.18. *For each j and m we have*

$$\left\|\partial_t^j(\mathfrak{D}^*_{\omega_{k,1}(t)}\mathfrak{D}_{\omega_{k,1}(t)} - \mathfrak{D}^*_{\omega_{k,1,\infty}}\mathfrak{D}_{\omega_{k,1,\infty}}(\phi))\right\|_{C^m(g_{k,1\infty})} \leq C\|\phi\|_{C^m(g_{k,1\infty})}/\sqrt{t},$$

where the constant C is independent of k.

Proof. By equation 2.7, we have

$$\begin{aligned}
&\left\|\partial_t^j(\mathfrak{D}^*_{\omega_{k,1}(t)}\mathfrak{D}_{\omega_{k,1}(t)} - \mathfrak{D}^*_{\omega_{k,1,\infty}}\mathfrak{D}_{\omega_{k,1,\infty}}(\phi))\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & \left\|\partial_t^j\left(\Delta^2_{\omega_{k,1}(t)} - \Delta^2_{\omega_{k,1,\infty}}\right)(\phi)\right\|_{C^m(g_{k,1\infty})} \\
&+C\left\|\partial_t^j\left(Scal\left(\omega_{k,1}(t)\right)\Delta_{\omega_{k,1}(t)}\left(\phi_t+\phi_\infty\right) - Scal\left(\omega_{k,1,\infty}\right)\Delta_{\omega_{k,1,\infty}}\left(\phi_\infty\right)\right)\right\|_{C^m(g_{k,1\infty})} \\
&+C\left\|\partial_t^j\left(\frac{\omega_{k,1}(t)^{r-2}\wedge\rho_{\omega_{k,1}(t)}\wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi)}{(\omega_{k,1}(t))^r} - \frac{(\omega_{k,1,\infty})^{r-2}\wedge\rho_{\omega_{k,1,\infty}}\wedge i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi)}{(\omega_{k,1,\infty})^r}\right)\right\|_{C^m(g_{k,1\infty})}.
\end{aligned}$$

Applying exactly the same argument as in the previous lemma and using the fact that $\Delta_{\omega_{k,1}(t)}\phi$ and all of its time derivatives converge smoothly, and so in particular have uniformly bounded operator norm, we estimate:

$$\begin{aligned}
&\left\|\partial_t^j\left(\Delta^2_{\omega_{k,1}(t)} - \Delta^2_{\omega_{k,1,\infty}}\right)(\phi)\right\|_{C^m(g_{k,1\infty})} \\
=\ & \left\|\partial_t^j\left(\Delta_{\omega_{k,1}(t)}\left(\Delta_{\omega_{k,1}(t)}\phi\right) - \Delta_{\omega_{k,1,\infty}}\Delta_{\omega_{k,1,\infty}}(\phi)\right)\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & \left\|\partial_t^j\left(\left(\Delta_{\omega_{k,1}(t)} - \Delta_{\omega_{k,1,\infty}}\right)\Delta_{\omega_{k,1}(t)}\phi\right)\right\|_{C^m(g_{k,1\infty})} + \left\|\partial_t^j\left(\Delta_{\omega_{k,1,\infty}}\left(\Delta_{\omega_{k,1}(t)}\phi\right) - \Delta_{\omega_{k,1,\infty}}(\phi)\right)\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & C\sum_{i=0}^{j}\left(\left\|\partial_t^i\left(\omega_{k,1}(t) - (\omega_{k,1,\infty})\right)\right\|_{C^m((g_{k,1,\infty}))} + \left\|\partial_t^i a_t^{0,1}\right\|_{C^m((g_{k,1,\infty}))}\right)\|\phi\|_{C^m(g_{k,1\infty})} \\
&+C\sum_{i=0}^{j}\left(\left\|\partial_t^i a_t^{1,0}\right\|_{C^m((g_{k,1,\infty}))} + \left\|\partial_t^i a_t^{1,0}\right\|_{C^m((g_{k,1,\infty}))}\cdot\left\|\partial_t^i a_t^{0,1}\right\|_{C^m((g_{k,1,\infty}))}\right)\|\phi\|_{C^m(g_{k,1\infty})} \\
\leq\ & C\|\phi\|_{C^m(g_{k,1\infty})}/\sqrt{t},
\end{aligned}$$

where we have again applied lemma 4.16.

Similarly

$$\begin{aligned}
&\left\|\partial_t^j\left(\frac{\omega_{k,1}(t)^{r-2}\wedge\rho_{\omega_{k,1}(t)}\wedge i\overline{\partial}_{J_t}\partial_{J_t}(\phi)}{(\omega_{k,1}(t))^r} - \frac{(\omega_{k,1,\infty})^{r-2}\wedge\rho_{\omega_{k,1,\infty}}\wedge i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi)}{(\omega_{k,1,\infty})^r}\right)\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & C\left(\sum_{i=0}^{j}\left\|\partial_t^i\left(\omega_{k,1}(t) - (\omega_{k,1,\infty})\right)\right\|_{C^m((g_{k,1,\infty}))} + \left\|\partial_t^i a_t^{0,1}\right\|_{C^m((g_{k,1,\infty}))} + \left\|\partial_t^i\left(\rho_{\omega_{k,1}(t)} - \rho_{\omega_{k,1,\infty}}\right)\right\|_{C^m((g_{k,1,\infty}))}\right) \\
&\times\|\phi\|_{C^m(g_{k,1\infty})}
\end{aligned}$$

$$
\begin{aligned}
& +C\left(\sum_{i=0}^{j}\left\|\partial_t^i a_t^{1,0}\right\|_{C^m\left(\left(g_{k,1,\infty}\right)\right)}+\left\|\partial_t^i a_t^{1,0}\right\|_{C^m\left(\left(g_{k,1,\infty}\right)\right)}\cdot\left\|\partial_t^i a_t^{0,1}\right\|_{C^m\left(\left(g_{k,1,\infty}\right)\right)}\right)\|\phi\|_{C^m(g_{k,1\infty})} \\
\leq\ & C\,\|\phi\|_{C^m(g_{k,1\infty})}/\sqrt{t},
\end{aligned}
$$

where the estimate on $\left\|\partial_t^i\left(\rho_{\omega_{k,1}(t)}-\rho_{\omega_{k,1,\infty}}\right)\right\|_{C^m\left(\left(g_{k,1,\infty}\right)\right)}$ in terms of the other quantities appearing in the above formula may be performed in exactly the same way as in Lemma 2.7, and where we use Lemma 4.14 to conclude that the resulting constant does not depend on k.

Finally,

$$
\begin{aligned}
& \left\|\partial_t^j\left(\left(Scal\left(\omega_{k,1}(t)\right)\Delta_{\omega_{k,1}(t)}(\phi)-Scal\left(\omega_{k,1,\infty}\right)\right)\Delta_{\omega_{k,1,\infty}}(\phi)\right)\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & \left\|\partial_t^j\left(\left(Scal\left(\omega_{k,1}(t)\right)-Scal\left(\omega_{k,1,\infty}\right)\right)\Delta_{\omega_{k,1}(t)}(\phi)\right)\right\|_{C^m(g_{k,1\infty})} \\
& +\left\|\partial_t^j\left(Scal\left(\omega_{k,1,\infty}\right)\left(\Delta_{\omega_{k,1}(t)}(\phi)-\Delta_{\omega_{k,1,\infty}}(\phi_\infty)\right)\right)\right\|_{C^m(g_{k,1\infty})} \\
\leq\ & C\left\|\partial_t^j\left(\left(Scal\left(\omega_{k,1}(t)\right)-Scal\left(\omega_{k,1,\infty}\right)\right)\right)\right\|_{C^m(g_{k,1\infty})}\|\phi\|_{C^m(g_{k,1\infty})} \\
& +C\left\|\partial_t^j\left(\left(\Delta_{\omega_{k,1}(t)}(\phi)-\Delta_{\omega_{k,1,\infty}}(\phi_\infty)\right)\right)\right\|_{C^m(g_{k,1\infty})}\|\phi\|_{C^m(g_{k,1\infty})} \\
\leq\ & C/\sqrt{t},
\end{aligned}
$$

by the previous lemma and where again the difference of the scalar curvatures may be computed as in Lemma 2.7 for large values of t. ☐

Lemma 4.19. *There exist $c>0, K>0$, such that for all p,ε and q as in Lemma2.7 and every $(\chi(t),\chi_\infty)\in W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})$ with*

$$
\begin{aligned}
& \|(\chi(t),\chi_\infty)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})} \\
=\ & \|(\chi(t)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}+\|\chi_\infty\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\leq c,
\end{aligned}
$$

the operators

$$\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,1}(t)}-Scal_{\omega_{k,1,\infty}}\right): W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})\to W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty}),$$

have a uniform bound

$$\left\|\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,1}(t)}-Scal_{\omega_{k,1,\infty}}\right)\right\|\leq C\left(\|(\chi(t)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}+\|\chi_\infty\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\right)$$

for the operator norm.

Proof. Applying Lemma and regrouping terms strategically, we may write

$$
\begin{aligned}
& \left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,1}(t)}-Scal_{\omega_{k,1,\infty}}\right)(\phi_t,\phi_\infty) \\
=\ & \left(\Delta^2_{\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)}-\Delta^2_{\omega_{k,1}(t)}\right)(\phi_t) \\
& +\left(\Delta^2_{\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)}-\Delta^2_{\omega_{k,1}(t)}+\Delta^2_{\omega_{k,1,\infty}}-\Delta^2_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}\right)(\phi_\infty) \\
& +\left(Scal\left(\omega_{k,1,\infty}\right)-Scal\left(\omega_{k,1}(t)\right)\right)\left(\Delta_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}(\phi_\infty)-\Delta_{\omega_{k,1,\infty}}(\phi_\infty)\right)
\end{aligned}
$$

$$+Scal\left(\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)\right)\left(\Delta_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}(\phi_\infty)-\Delta_{\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)}(\phi_t+\phi_\infty)\right)$$

$$+Scal\left(\omega_{k,1}(t)\right)\left(\Delta_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}(\phi_\infty)-\Delta_{\omega_{k,1,\infty}}(\phi_\infty)+\Delta_{\omega_{k,1}(t)}(\phi_t+\phi_\infty)-\Delta_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}(\phi_\infty)$$

$$+\frac{\frac{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}{\left(\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)\right)^r}\left(\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)\right)^{r-2}\wedge\rho_{\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)}\wedge i\overline{\partial}_{J_t}\partial_{J_t}\left(\phi_t+\phi_\infty\right)}{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}$$

$$-\frac{\frac{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}{\left(\omega_{k,1}(t)\right)^r}\left(\omega_{k,1}(t)\right)^{r-2}\wedge\rho_{\omega_{k,1}(t)}\wedge i\overline{\partial}_{J_t}\partial_{J_t}\left(\phi_t+\phi_\infty\right)}{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}$$

$$+\frac{\frac{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}{\left(\omega_{k,1,\infty}\right)^r}(\omega_{k,1,\infty})^{r-2}\wedge\rho_{\omega_{k,1,\infty}}\wedge i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)}{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}$$

$$-\frac{(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty))^{r-2}\wedge\rho_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}\wedge i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)}{\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)^r}$$

$$+\left(Scal\left(\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)\right)-Scal\left(\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)\right)+Scal\left(\omega_{k,1}(t)\right)-Scal\left(\omega_{k,1,\infty}\right)\right)\Delta_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}}$$

Arguing in exactly the same way as in the proofs of Lemmas 4.17 and 4.10 above, we conclude that there is a bound

$$\begin{aligned}
&\left\|\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,1}(t)}-Scal_{\omega_{k,1,\infty}}\right)(\phi_t,\phi_\infty)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
\leq\;& C\left\|(\chi(t),\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}\left\|(\phi_t,\phi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}\\
&+C\left\|\left(Scal\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)-Scal\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)+Scal\left(\omega_{k,1}(t)\right)-Scal\left(\omega_{k,1,\infty}\right)\right)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
&\times\left\|\phi_\infty\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\\
&+C\left\|\rho_{\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)}-\rho_{\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)}+\rho_{\omega_{k,1,\infty}}-\rho_{\omega_{k,1}(t)}\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
&\times\left\|(\phi_t,\phi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}.
\end{aligned}$$

To bound the last two terms, we generalise the argument of Lemmas 2.7-2.10 of [F] If we write $g_{k,1}(\chi_t+\chi_\infty)$, $g_{k,1}(\chi_\infty)$, $g_{k,1}(t)$, and $g_{k,1,\infty}$ for the metrics corresponding to $\omega_{k,1}(t)+i\overline{\partial}_{J_t}\partial_{J_t}(\chi_t+\chi_\infty)$, $\omega_{k,1,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\chi_\infty)$, $\omega_{k,1}(t)$, and $\omega_{k,1,\infty}$, we may write

$$\begin{aligned}
g_{k,1}(\chi_t+\chi_\infty) &= g_{k,1}(t)+h_{k,1}(\chi_t+\chi_\infty)\\
g_{k,1}(\chi_t+\chi_\infty) &= g_{k,1}(\chi_\infty)+H_{k,1}(\chi_t+\chi_\infty)\\
g_{k,1}(\chi_\infty) &= g_{k,1,\infty}+h_{k,1}(\chi_\infty)\\
g_{k,1}(t) &= g_{k,1,\infty}+H_{k,1}(t)
\end{aligned}$$

for some symmetric two-tensors $h_{k,1}(\chi_t+\chi_\infty)$, $h_{k,1}(\chi_\infty)$, $H_{k,1}(\chi_t+\chi_\infty)$,and $H_{k,1}(t)$, where

$$H_{k,1}(\chi_t+\chi_\infty),H_{k,1}(t)\in W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty}),$$

and

$$\begin{aligned}
\left\|H_{k,1}(\chi_t+\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq C\left\|(\chi(t),\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})},\\
\left\|h_{k,1}(\chi_t+\chi_\infty)-h_{k,1}(\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq C\left\|(\chi(t),\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})},
\end{aligned}$$

$$\|h_{k,1}(\chi_\infty)\|_{L^2_{4(p+1)}(g_{k,1,\infty})} \quad \leq \quad \|\chi_\infty\|_{L^2_{4(p+1)}(g_{k,1,\infty})}$$

may be made arbitrarily small by making

$$\|(\chi(t),\chi_\infty)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}$$

arbitrarily small. For the respective Levi-Civita connections, we have

$$\begin{aligned}
\nabla_{g_{k,1}(\chi_t+\chi_\infty)} &= \nabla_{g_{k,1}(t)} + a_{k,1}\left(\chi_t+\chi_\infty\right),\\
\nabla_{g_{k,1}(\chi_t+\chi_\infty)} &= \nabla_{g_{k,1}(\chi_\infty)} + b_{k,1}\left(\chi_t+\chi_\infty\right),\\
\nabla_{g_{k,1}(\chi_\infty)} &= \nabla_{g_{k,1,\infty}} + a_{k,1}(\chi_\infty),\\
\nabla_{g_{k,1}(t)} &= \nabla_{g_{k,1,\infty}} + b_{k,1}(t)
\end{aligned}$$

where

$$\begin{aligned}
a_{k,1}\left(\chi_t+\chi_\infty\right)\cdot\left(g_{k,1}(t)+h_{k,1}(\chi_t+\chi_\infty)\right) &= -\nabla_{g_{k,1}(t)}\left(h_{k,1}(\chi_t+\chi_\infty)\right),\\
a_{k,1}(\chi_\infty)\cdot\left(g_{k,1,\infty}+h_{k,1}(\chi_\infty)\right) &= -\nabla_{g_{k,1,\infty}}\left(h_{k,1}(\chi_\infty)\right),\\
b_{k,1}\left(\chi_t+\chi_\infty\right)\cdot\left(g_{k,1}(\chi_\infty)+H_{k,1}(\chi_t+\chi_\infty)\right) &= -\nabla_{g_{k,1}(\chi_\infty)}\left(H_{k,1}(\chi_t+\chi_\infty)\right)\\
b_{k,1}(t)\cdot\left(g_{k,1,\infty}+H_{k,1}(t)\right) &= -\nabla_{g_{k,1,\infty}}\left(H_{k,1}(t)\right)
\end{aligned}$$

where $\cdot$ is the algebraic operation carrying out the identification $T^*X\otimes End(T^*X)\simeq (T^*X)^{\otimes 3}$.

From the above definitions, we obtain that

$$\begin{aligned}
g_{k,1}(\chi_t+\chi_\infty) &= g_{k,1,\infty}+H_{k,1}(\chi_t+\chi_\infty)+h_{k,1}(\chi_t+\chi_\infty),\\
g_{k,1}(\chi_t+\chi_\infty) &= g_{k,1,\infty}+H_{k,1}(t)+h_{k,1}(\chi_\infty),
\end{aligned}$$

$$\begin{aligned}
\nabla_{g_{k,1}(\chi_t+\chi_\infty)} &= \nabla_{g_{k,1}(t)}+a_{k,1}\left(\chi_t+\chi_\infty\right)=\nabla_{g_{k,1,\infty}}+b_{k,1}(t)+a_{k,1}\left(\chi_t+\chi_\infty\right),\\
\nabla_{g_{k,1}(\chi_t+\chi_\infty)} &= \nabla_{g_{k,1}(\chi_\infty)}+b_{k,1}\left(\chi_t+\chi_\infty\right)=\nabla_{g_{k,1,\infty}}+a_{k,1}(\chi_\infty)+b_{k,1}\left(\chi_t+\chi_\infty\right),
\end{aligned}$$

and therefore in particular

$$b_{k,1}\left(\chi_t+\chi_\infty\right)=b_{k,1}(t)+a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty).$$

Then for the curvatures of the various metrics, we calculate

$$\begin{aligned}
& R(g_{k,1}(\chi_t+\chi_\infty))-R\left(g_{k,1}(t)\right)+R\left(g_{k,1,\infty}\right)-R\left(g_{k,1}(\chi_\infty)\right)\\
= \;& R\left(g_{k,1,\infty}\right)+\nabla_{g_{k,1,\infty}}\left(b_{k,1}(t)+a_{k,1}\left(\chi_t+\chi_\infty\right)\right)\\
& +\left(b_{k,1}(t)+a_{k,1}\left(\chi_t+\chi_\infty\right)\right)\wedge\left(b_{k,1}(t)+a_{k,1}\left(\chi_t+\chi_\infty\right)\right)\\
& -R\left(g_{k,1,\infty}\right)-\nabla_{g_{k,1,\infty}}\left(b_{k,1}(t)\right)-b_{k,1}(t)\wedge b_{k,1}(t)\\
& +R\left(g_{k,1,\infty}\right)-R\left(g_{k,1,\infty}\right)-\nabla_{g_{k,1,\infty}}a_{k,1}(\chi_\infty)-a_{k,1}(\chi_\infty)\wedge a_{k,1}(\chi_\infty)\\
= \;& \nabla_{g_{k,1,\infty}}\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\\
& +a_{k,1}\left(\chi_t+\chi_\infty\right)\wedge b_{k,1}(t)+b_{k,1}(t)\wedge a_{k,1}\left(\chi_t+\chi_\infty\right)\\
& +a_{k,1}\left(\chi_t+\chi_\infty\right)\wedge a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\wedge a_{k,1}(\chi_\infty)\\
= \;& \nabla_{g_{k,1,\infty}}\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\\
& +\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\wedge b_{k,1}(t)+b_{k,1}(t)\wedge\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\\
& +\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\wedge\left(a_{k,1}\left(\chi_t+\chi_\infty\right)\right)+a_{k,1}(\chi_\infty)\wedge\left(a_{k,1}\left(\chi_t+\chi_\infty\right)-a_{k,1}(\chi_\infty)\right)\\
& +a_{k,1}(\chi_\infty)\wedge b_{k,1}(t)+b_{k,1}(t)\wedge a_{k,1}(\chi_\infty)
\end{aligned}$$

Then we obtain

$$\begin{aligned}&\left\|R(g_{k,1}(\chi_t+\chi_\infty)) - R\left(g_{k,1}(t)\right) + R\left(g_{k,1,\infty}\right) - R\left(g_{k,1}(\chi_\infty)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\\ \leq\ & C\left(\left\|a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|a_{k,1}(\chi_\infty)\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\right),\end{aligned}$$

where by Lemma 4.14, the constant is independent of k.

In order to estimate these quantities we write:

$$\begin{aligned}& a_{k,1}\left(\chi_t+\chi_\infty\right)\cdot\left(g_{k,1,\infty}+H_{k,1}(t)+h_{k,1}(\chi_t+\chi_\infty)\right) - a_{k,1}(\chi_\infty)\cdot\left(g_{k,1,\infty}+h_{k,1}(\chi_\infty)\right)\\ =\ & \nabla_{g_{k,1,\infty}}\left(h_{k,1}(\chi_\infty) - H_{k,1}(t) - h_{k,1}(\chi_t+\chi_\infty)\right),\end{aligned}$$

so that

$$\begin{aligned}& \left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\cdot g_{k,1,\infty}\\ =\ & \nabla_{g_{k,1,\infty}}\left(h_{k,1}(\chi_\infty) - H_{k,1}(t) - h_{k,1}(\chi_t+\chi_\infty)\right)\\ & + a_{k,1}\left(\chi_t+\chi_\infty\right)\cdot\left(H_{k,1}(t) + h_{k,1}(\chi_t+\chi_\infty) - h_{k,1}(\chi_\infty)\right)\\ & + \left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\cdot h_{k,1}(\chi_\infty)\\ =\ & \nabla_{g_{k,1,\infty}}\left(2\left(h_{k,1}(\chi_\infty) - h_{k,1}(\chi_t+\chi_\infty)\right) - H_{k,1}(\chi_t+\chi_\infty)\right)\\ & + a_{k,1}\left(\chi_t+\chi_\infty\right)\cdot\left(H_{k,1}(\chi_t+\chi_\infty) + 2\left(h_{k,1}(\chi_t+\chi_\infty) - h_{k,1}(\chi_\infty)\right)\right)\\ & + \left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\cdot h_{k,1}(\chi_\infty),\end{aligned}$$

and so

$$\begin{aligned}& \left\|\left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\\ =\ & \left\|\left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\cdot g_{k,1,\infty}\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\\ \leq\ & C_1\left(\left\|h_{k,1}(\chi_\infty) - h_{k,1}(\chi_t+\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|H_{k,1}(\chi_t+\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\right)\\ & + C_2\left\|h_{k,1}(\chi_\infty)\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\left\|\left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})},\end{aligned}$$

and so for

$$\begin{aligned}& \left\|\left(a_{k,1}\left(\chi_t+\chi_\infty\right) - a_{k,1}(\chi_\infty)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\\ \leq\ & C\frac{\left(\left\|h_{k,1}(\chi_\infty) - h_{k,1}(\chi_t+\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|H_{k,1}(\chi_t+\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\right)}{\left(1-\frac{1}{2}c^{-1}\left\|h_{k,1}(\chi_\infty)\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})}\right)},\\ \leq\ & C\left\|(\chi(t),\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}\end{aligned}$$

where we have taken $c < \frac{1}{2}C_2^{-1}$. In a completely analogous way we achieve a bound

$$\left\|a_{k,1}(\chi_\infty)\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})} \leq \left\|\chi_\infty\right\|_{L^2_{4(p+1)}(g_{k,1,\infty})},$$

and so finally we get

$$\begin{aligned}&\left\|R(g_{k,1}(\chi_t+\chi_\infty)) - R\left(g_{k,1}(t)\right) + R\left(g_{k,1,\infty}\right) - R\left(g_{k,1}(\chi_\infty)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})}\\ \leq\ & \left\|(\chi(t),\chi_\infty)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})}.\end{aligned}$$

Since the Ricci and scalar curvatures are contractions of the full curvature, the same estimate also applies to them, and we obtain the required bound. □

Later we will need the the following lemma, which says that the constant in the parabolic estimate 7.10 is independent of the parameter k.

Lemma 4.20. *Let $L_{s(t)}$ be a path of smooth self-adjoint elliptic operators of order 4 on $\mathbb{P}(E)$ converging smoothly to an operator L_∞. Suppose $\psi(s(t)) \in W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})$, and $\psi(s(t)) \perp \ker L_\infty$ for all s. Then there is a constant A, depending only on p, q, ε, such that for the solution $\phi(s(t)) \in W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})$, of the initial value problem*

$$\begin{aligned} \frac{\partial \phi(s(t))}{\partial s} + L_{s(t)}\left(\phi(s(t))\right) &= \psi(s(t)) \\ \phi(0) &= \phi_0 \end{aligned}$$

and sufficiently large k, we have an estimate:

$$\begin{aligned} &\|(\phi(s(t))\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})} \\ \leq\ & A\left(\|\phi_0\|_{L^2_{4p+2}} + \|\psi(s(t))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}))}\right). \end{aligned}$$

The constant A is in particular independent of k.

5. The Approximate solutions

5.1. **The approximation theorem.** We fix the bundle $(E,h) \to (\Sigma, \omega_\Sigma)$ and the projectivisation $\mathbb{P}(E)$, as in the preceding sections, but from here on out we require ω_Σ to be a metric of constant scalar curvature. We will at times move the hermitian metric h on the bundle E, and other times we will need to move the holomorphic structure $\overline{\partial}_E$, which is tantamount to moving the operator $\overline{\partial}_J$ on $\mathbb{P}(E)$. In the second case, we will need to consider various one-parameter families of geometric objects $\sigma(t)$, associated to this moving family $\overline{\partial}_{J_t}$. These families will always converge smoothly. Then if we write $|\sigma(t)|$ for the pointwise, the notation

$$\sigma(t) = \mathcal{O}(k^{-l})$$

means that there is a constant C independent of t such that

$$|\sigma(t)| \leq Ck^{-l}.$$

In case we move the hermitian metric instead, we will obtain one parameter families $\widehat{\sigma}(t) = \widetilde{g}_t^*(\sigma(t))$. Since the $\widetilde{g}_t$ are not converging, these families will not converge, but by a small abuse of notation, when we write

$$\widehat{\sigma}(t) = \mathcal{O}(k^{-l}),$$

we will simply mean that

$$(\widetilde{g}_t^{-1})^*(\widehat{\sigma}(t)) = \sigma(t)$$

has the above stated property.

With this in mind, the main goal of this section is to write down a formal approximate solution to the Calabi flow equation in the sense that for a sufficiently large choice of an auxillary parameter k, we will produce for each l a path of Kahler metrics $\omega_{k,l}(s(t))$ (compatible with the holomorphic structure $J_{s(t)}$ where $s = t \cdot r/k$) such that

$$\frac{\partial \tilde{g}^*_{s(t)}(\omega_{k,l}(s(t)))}{\partial t} + i\bar{\partial}\partial Scal\left(\tilde{g}^*_{s(t)}(\omega_{k,l}(s(t)))\right) = \mathcal{O}(k^{-(l+1)}).$$

More specifically, we will prove the following theorem.

Theorem 5.1. *Let $(E,h) \to (\Sigma, \omega_\Sigma)$ be an Hermitian vector bundle over a Riemann surface, equipped with a constant scalar curvature metric. Fix a smooth connection $A = A_0$ which is the Chern connection $(\overline{\partial}_E, h)$ for holomorphic structure giving rise to holomorphic vector bundle $\mathcal{E} = \mathcal{E}_0$. We assume that this holomorphic structure is simple, and has the property that the associated graded object of its Harder-Narasimhan filtration contains only stable factors.*

Fix $k >> 0$. For each $l \geq 1$, and for any fixed number $S \in [0,\infty)$, there is a path $\eta_s^S \in \mathfrak{su}(E)$, and paths of Kähler potentials $\Theta_{k,m}(t) \in \pi^* C^\infty(\Sigma)$, $\Xi_{k,m}(\eta_s^S) \in C^\infty(\mathbb{P}(E))$, and $\Omega_{k,m}(t) \in C^\infty_h(\mathbb{P}(E))_\perp$, on $\mathbb{P}(E)$, and smooth, functions $\Theta_{k,m,\infty}$, $\Xi_{k,m,\infty}$, and $\Omega_{k,m,\infty}$ such that for each k and l and every $1 \leq m \leq l$, and all p, q, and ε, the following hold

- *The paths of functions $\Theta_{k,m}(t)$, $\Xi_{k,m}(\eta_s^S)$, and $\Omega_{k,m}(t)$ converge to $\Theta_{k,m,\infty}$, $\Xi_{k,m,\infty}$, and $\Omega_{k,l,\infty}$ respectively in $C^\infty(X)$ as $t \to \infty$. Furthermore, if we define*

$$\begin{aligned}(5.1)\quad & \omega^S_{k,l}(s(t)) \\ &= \omega(h, J_s) + k\pi^*\omega_\Sigma + i\bar{\partial}_{J_s}\partial_{J_s}\left(\sum_{m=1}^{l-1} k^{-m+1}\Theta_m(t) + \sum_{m=1}^{l-1} k^{-m}\Xi_{k,m}(\eta_s^S) + \sum_{m=1}^{l-1} k^{-(m+1)}\Omega_m(t)\right),\end{aligned}$$

where $s = t \cdot r/k$, then $\omega_{k,l}(t)$ converges smoothly to a Kähler metric

$$\begin{aligned}(5.2)\quad & \omega_{k,l,\infty} \\ &= \omega(h, J_\infty) + k\pi^*\omega_\Sigma + i\bar{\partial}_{J_\infty}\partial_{J_\infty}\left(\sum_{m=1}^{l-1} k^{-m+1}\Theta_{k,m,\infty} + \sum_{m=1}^{l-1} k^{-m}\Xi_{k,m,\infty} + \sum_{m=1}^{l-1} k^{-(m+1)}\Omega_{k,m,\infty}\right),\end{aligned}$$

where J_∞ is the holomorphic structure corresponding to the manifold $\mathbb{P}(\mathcal{E}_\infty) = \mathbb{P}(Gr\mathcal{E})$, arising from the limit of the Yang-Mills flow.

- *Writing $\omega_{k,l}(s(t))$ for $\omega^S_{k,l}(s(t))$, for each l there exists a path $H(\omega_{k,l}(s))$ of smooth functions such that if V_s is the time dependent infinitesimal generator associated to $\widetilde{g}_s$, for all $s \in [0,S]$*

$$rk^{-1}\left(\frac{\partial \omega_{k,l}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,l}(s)\right) = i\bar{\partial}_{J_s}\partial_{J_s} H(\omega_{k,l}(s)). \tag{5.3}$$

- *Moreover,*

$$\begin{aligned} & Scal\,(\omega_{k,l}(s(t))) + H(\omega_{k,l}(s(t))) \\ = \;& Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}\left(Scal\,(\omega_\Sigma)\right) \\ + & \sum_{M=l+1} k^{-M}(\Psi^{(l)}_{\Sigma,M}(s) + \Psi^{(l)}_{\Phi_h,M}(s) + \Psi^{(l)}_{\perp,l}(s)) \\ = \;& \mathcal{O}(k^{-(l+1)}),\end{aligned} \tag{5.4}$$

so that in particular

$$i\bar{\partial}_{J_s}\partial_{J_s}\left(Scal\,(\omega_{k,l}\,(s)) + H\left(\omega_{k,l}(s)\right)\right) = \mathcal{O}\left(k^{-(l+1)}\right).$$

- *There is a smooth function $H(\omega_{k,l,\infty})$ such that*

$$H(\omega_{k,l}(s)) \overset{C^\infty}{\to} H(\omega_{k,l,\infty}) \tag{5.5}$$

and we also have

$$Scal(\omega_{k,l}(s(t))) \overset{C^\infty}{\to} Scal(\omega_{k,l,\infty})$$

so that in particular

$$
\begin{aligned}
Scal\,(\omega_{k,l,\infty}) + H(\omega_{k,l,\infty}) &= Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}\left(Scal\,(\omega_\Sigma)\right) \\
&+ \sum_{M=l+1} k^{-M}(\Psi^{(l)}_{\Sigma,M,\infty} + \Psi^{(l)}_{\Phi_h,M,\infty} + \Psi^{(l)}_{\perp,l,\infty}(s)) \\
&= \mathcal{O}(k^{-(l+1)}),
\end{aligned} \tag{5.6}
$$

and

$$
i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(Scal\,(\omega_{k,l,\infty}) + H\left(\omega_{k,l,\infty}\right)\right) = \mathcal{O}\left(k^{-(l+1)}\right). \tag{5.7}
$$

- *The path of Kähler metrics $\hat{\omega}_{k,l}(s(t)) = g_s^*\left(\omega_{k,l}(s(t))\right)$ on the fixed complex manifold $\mathbb{P}(\mathcal{E})$ formally solves solves*

$$
\frac{\partial\hat{\omega}_{k,l}(s(t))}{\partial t} + i\bar{\partial}_J\partial_J Scal\left(\hat{\omega}_{k,l}(s(t))\right) = \mathcal{O}(k^{-(l+1)}), \tag{5.8}
$$

for $s \in [0, S]$.

- *Finally there are estimates of the form*

$$
\begin{aligned}
\left\|\Psi^{(l)}_{\Sigma,M}(s) - \Psi^{(l)}_{\Sigma,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\
\left\|\Psi^{(l)}_{\Phi_h,M}(s) - \Psi^{(l)}_{\Phi_h,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\
\left\|\Psi^{(l)}_{\perp,M}(s)) - \Psi^{(l)}_{\perp,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}),
\end{aligned} \tag{5.9}
$$

for all $M \geq l+1$, so that in particular

$$
\|Scal(\omega_{k,l}(s)) + H(\omega_{k,l}(s)) - (Scal(\omega_{k,l,\infty}) + H(\omega_{k,l,\infty}))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} \leq Ck^{-(l+1/2)} \tag{5.10}
$$

for all $p, q,$ and ε.

- *In fact, the same estimate is true using the metric $g_{k,l,\infty}$ instead of $g_{k,1,\infty}$, that is:*

$$
\begin{aligned}
&\|Scal\,(\omega_{k,l}(s(t))) + H(\omega_{k,l}(s(t)) - (Scal\,(\omega_{k,l,\infty}) + H(\omega_{k,l,\infty}))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})} \\
= \;& \mathcal{O}(k^{-(l+1/2)}),
\end{aligned} \tag{5.11}
$$

for all l.

Remark 5.2. The parameter S appears in the above theorem because at a certain point in the proof we will have to introduce a cutoff function supported in the interval $[0, 2S]$, where the choice of S is arbitrary. The theorem gives an entire one parameter family of paths of metrics $\omega^S_{k,l}(s(t))$, with each choice of S giving a different path. Mostly however, we will omit the S superscript, unless it is absolutely necessary.

5.2. **The scalar curvature expansion and the approximation to second order.** The proof of Theorem 5.1 will be by induction on l. The sequence of lemmas in this subsection will give the result for $l = 1$. Our ansatz for the metrics in Theorem 5.1 will be given by the family of two forms on $\mathbb{P}(E)$ associated to the path of connections A_s at time $s = t \cdot r/k$. Below we will sometimes write one parameter families of objects on $\mathbb{P}(E)$ as being functions of the variable $s(t)$ to emphasise the fact that they are functions of t and k. Define

$$
\omega_{k,1}(s(t)) = \omega(h, J_s) + k\omega_\Sigma \tag{5.12}
$$

and

$$\begin{aligned} & \hat{\omega}_{k,1}\left(s(t)\right) \\ (5.13) \qquad = \; & \tilde{g}_s^*\left(\omega_{k,1}(s)\right) = \omega(h_s, J) + k\omega_\Sigma. \end{aligned}$$

We will begin with a general lemma that holds for any path of connections with uniformly bounded Hermitian-Einstein tensor.

Lemma 5.3. *For any path A_t of connections on E (that is, $||\Lambda_{\omega_\Sigma} F_{A_t}||_{L^\infty} \leq C$), so that the associated two forms*

$$\omega_k(h, J_t) = \omega(h, J_t) + k\omega_\Sigma$$

are Kähler, the scalar curvature satisfies the following pointwise expansion in powers of k^{-1}:

$$\begin{aligned} & Scal(\omega_k(h, J_t)) \\ (5.14) \qquad = \; & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}(-2r\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_t}) + Scal(\omega_\Sigma)) \\ + \; & \sum_{l=2} k^{-l}(\Psi_{\Sigma,l}(t) + \Psi_{\Phi_h,l}(t) + \Psi_{\perp,l}(t)), \end{aligned}$$

where $\Lambda_{\omega_\Sigma} F^\circ_{A_t}$ is the trace-free part

$$\Lambda_{\omega_\Sigma} F^\circ_{A_t} = \Lambda_{\omega_\Sigma} F_{A_t} - \frac{tr\left(\Lambda_{\omega_\Sigma} F_{A_t}\right)}{r} Id_E$$

of the contracted curvature, and where

$$\Psi_{\Sigma,l}(t) + \Psi_{\Phi_h,l}(t) + \Psi_{\perp,l}(t) \in \pi^* C^\infty(\Sigma) \oplus \Phi_h(\mathfrak{su}(E)) \oplus C^\infty(\mathbb{P}(E))_\perp,$$

so that if we set

$$H\left(\omega_k\left(h, J_t\right)\right) = \frac{2r}{k}\left(\Phi_h\left(\Lambda_{\omega_\Sigma} F^\circ_{A_t}\right)\right), \tag{5.15}$$

then in particular

$$i\overline{\partial}_{J_t}\partial_{J_t}\left(Scal\left(\omega_k(h, J_t)\right) + H\left(\omega_k(h, J_t)\right)\right) = \mathcal{O}\left(k^{-2}\right). \tag{5.16}$$

Similarly, if A_t converges smoothly to a limit A_∞ giving rise to a (unique) limiting holomorphic structure J_∞, then for the limiting metric

$$\omega_k(h, J_\infty) = \omega(h, J_\infty) + k\omega_\Sigma$$

there is an expansion of the form

$$\begin{aligned} & Scal(\omega_k(h, J_\infty)) \\ (5.17) \qquad = \; & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}(-2r\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_\infty}) + Scal(\omega_\Sigma)) \\ + \; & \sum_{l=2} k^{-l}(\Psi_{\Sigma,l,\infty} + \Psi_{\Phi_h,l,\infty} + \Psi_{\perp,l,\infty}) \end{aligned}$$

so that if we set

$$H\left(\omega_k\left(h, J_\infty\right)\right) = \frac{2r}{k}\left(\Phi_h\left(\Lambda_{\omega_\Sigma} F^\circ_{A_\infty}\right)\right) \tag{5.18}$$

$$i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(Scal\left(\omega_k\left(h, J_\infty\right)\right) + H\left(\omega_k\left(h, J_\infty\right)\right)\right) = \mathcal{O}\left(k^{-2}\right).$$

Proof. In order to calculate the scalar curvature, we will first calculate their Ricci forms and then take a trace. Recall from Section 2.1 that the hermitian metrics $g_{k,1}(t) = g_k(h, J_t) \in \Gamma\left(T^*\mathbb{P}(\mathcal{E}_t) \otimes \overline{T^*\mathbb{P}(\mathcal{E}_t)}\right)$ (associated with the Kähler forms $\omega_{k,1}(t) = \omega_k(h, J_t)$) on $\mathbb{P}(\mathcal{E}_t)$ induce the Hermitian metric $\frac{(\omega_{k,1}(t))^r}{r!} \in \Gamma(K_{\mathbb{P}(\mathcal{E}_t)} \otimes \overline{K_{\mathbb{P}(\mathcal{E}_t)}})$ on $K^*_{\mathbb{P}(\mathcal{E}_t)} = \det(T\mathbb{P}(\mathcal{E}_t))$. Moreover, and the Ricci curvatures $\mathrm{Ric}(\omega_{k,1}(t))$ are given by $iF_{K^*_{\mathbb{P}(\mathcal{E}_t)}}(\frac{(\omega_{k,1}(t))^r}{r!})$, the curvature of this induced metric on the anti-canonical bundle.

By definition

$$\omega_{k,1}(t) = \omega_{FS}(h, J_t) + (\Phi_h^*(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma.$$

Since $\omega_{FS}(h, J_t)$ and $(\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\omega_\Sigma$ are positive definite on $\mathcal{V}_t$ and $\mathcal{H}_t$ respectively, they define Hermitian metrics on these bundles, and therefore the forms

$$\frac{(\omega_{FS}(h, J_t))^{r-1}}{(r-1)!} \in \Gamma(\det \mathcal{V}_t^* \otimes \det \overline{\mathcal{V}}_t^*) \text{ and } (\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\omega_\Sigma \in \Gamma(\mathcal{H}_t^* \otimes \overline{\mathcal{H}}_t^*)$$

are the induced Hermitian metrics on $\Lambda_{\mathbb{C}}^{r-1}\mathcal{V}_t = \det(\mathcal{V}_t)$ and $\det(\mathcal{H}_t) = \mathcal{H}_t$. We may decompose the curvature $iF_{K^*_{\mathbb{P}(\mathcal{E}_t)}}(\frac{(\omega_{k,1}(t))^r}{r!})$ into the curvatures $iF_{\det(\mathcal{V}_t)}(\frac{(\omega_{FS}(h,J)(t))^{r-1}}{(r-1)!})$ and $iF_{\mathcal{H}_t}((\Phi_h(-\Lambda_{\omega_\Sigma} F_A) + k)\, \omega_\Sigma)$, of these induced metrics.

Namely, from the exact sequence

$$0 \longrightarrow \mathcal{V}_t \longrightarrow T\mathbb{P}(\mathcal{E}_t) \longrightarrow \mathcal{H}_t \longrightarrow 0,$$

and the decomposition of $\omega_{k,1}(t)$ there is a smooth, metric splitting

$$(T\mathbb{P}(\mathcal{E}_t), \omega_{k,1}(t)) = (\mathcal{V}_t, r\omega_{FS}(h, J_t)) \oplus (\mathcal{H}_t, (\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma)$$

and taking determinants this gives an isometric isomorphism

$$\left(K^*_{\mathbb{P}(\mathcal{E}_t)}, \frac{(\omega_k\,(h, J_t))^r}{r!}\right) \simeq \left(det\mathcal{V}_t, \frac{(\omega_{FS}(h, J_t))^{r-1}}{(r-1)!}\right) \otimes (\mathcal{H}_t, (\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma)$$

since $\mathcal{H}_t$ is a line bundle. Therefore

$$\begin{aligned}
\rho_k(\omega_{k,1}(t)) &= iF_{K^*_{\mathbb{P}(\mathcal{E}_t)}} = i\bar{\partial}_{J_t}\partial_{J_t} log\left(\frac{((\Phi_h^*(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma + \omega_{FS}(h, J_t))^r}{r!}\right) \\
&= i\bar{\partial}_{J_t}\partial_{J_t} log\left((\Phi_h^*(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma \otimes \frac{(\omega_{FS}(h, J_t))^{r-1}}{(r-1)}\right) \\
&= i\bar{\partial}_{J_t}\partial_{J_t} log\left(\frac{(\omega_{FS}(h, J_t))^{r-1}}{(r-1)}\right) + i\bar{\partial}_{J_t}\partial_{J_t} log\left((\Phi_h(-\Lambda_{\omega_\Sigma} F_{A_t}) + k)\, \omega_\Sigma\right) \\
&= iF_{\det(\mathcal{V}_t)}\left(\frac{(\omega_{FS}(h, J)(t))^{r-1}}{(r-1)!}\right) + iF_{\mathcal{H}_t}\left((\Phi_h(-\Lambda_{\omega_\Sigma} F_A) + k)\, \omega_\Sigma\right)
\end{aligned}$$

To calculate $iF_{\det(\mathcal{V}_t)}(\frac{(\omega_{FS}(h,J)(t))^{r-1}}{(r-1)!})$, consider the Euler exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow (\mathcal{E}_t)_x \otimes \mathcal{O}_{\mathbb{P}((\mathcal{E}_t)_x)}(1) \longrightarrow T\mathbb{P}((\mathcal{E}_t)_x) \longrightarrow 0$$

which globalises to give an exact sequence

$$0 \to \mathbb{C} \to \pi^*\mathcal{E}_t \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_t)}(1) \to \mathcal{V}_t \to 0$$

over $\mathbb{P}\,(\mathcal{E}_t)$. Then this gives an isomorphism

$$det\mathcal{V}_t \cong \det \mathcal{E} \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_t)}(1))^{\otimes r}.$$

Under this isomorphism the metric $\frac{(\omega_{FS}(h,J)(t))^{r-1}}{(r-1)!}$ corresponds to the tensor product of $\det h$, with the metric induced by h (through its dual) on $\mathcal{O}_{\mathbb{P}(\mathcal{E}_t)}(1)$. Therefore

$$\begin{aligned} iF_{det\mathcal{V}_t} &= riF_{h_{\mathcal{L}_t}} = r\omega(h,J_t) + iF_{(\det h, \det \mathcal{E}_t)} \\ &= r\omega_{FS}(h,J_t) + r\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\omega_\Sigma + itr(F_{A_t}) \end{aligned}$$

We may also think of ω_Σ as giving a different metric on $\mathcal{H}_t$ whose curvature is exactly ρ_Σ, so we have that

$$\begin{aligned} iF_{\mathcal{H}_t} - \rho_\Sigma &= i\bar\partial_{J_t}\partial_{J_t} log\left(\frac{(\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma}{\omega_\Sigma}\right) \\ &= i\bar\partial_{J_t}\partial_{J_t} log\left(1 + k^{-1}\left(\Phi_h^*(-\Lambda_{\omega_\Sigma}F_{A_t})\right)\right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \rho_k(\omega_{k,1}(t)) &= iF_{\mathcal{V}_t\otimes\mathcal{H}_t} = iF_{det\mathcal{V}_t,h} + iF_{\mathcal{H}_t,h} \\ &= r\omega_{FS}(h,J_t) + r\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\omega_\Sigma + itr(F_{A_t}) + \rho_\Sigma \\ &\quad + i\bar\partial_{J_t}\partial_{J_t} log\left(1 + k^{-1}\left(\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\right)\right) \\ &= r\omega_{FS}(h,J_t) + r\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\omega_\Sigma + itr(F_{A_t}) + \rho_\Sigma \\ &\quad + \sum_{j=0}^{\infty}(-1)^j k^{-(j+1)} i\bar\partial_{J_t}\partial_{J_t}\left((\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}))^{j+1}\right), \end{aligned} \tag{5.19}$$

where we have used in the last line that $\log(1+x) = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{x^j}{j}$ for $|x| < 1$ (note that $k >> 0$).

Now the scalar curvature of $\omega_{k,1}(t)$ is by definition $\mathrm{Scal}(\omega_k(h,J_t)) = \Lambda_{\omega_{k,1}(t)}(\rho_k(\omega_{k,1}(t)))$. For $\gamma \in \Lambda^2(V^*)$ and $\beta \in H^*$ define the vertical and horizontal traces by

$$\Lambda_{\omega_{FS}(h,J_t)}(\gamma) = (r-1)\frac{\gamma \wedge (\omega_{FS}(h,J_t))^{r-2}}{(\omega_{FS}(h,J_t))^{r-1}} \text{ and } \Lambda_{\omega_\Sigma}(\beta) = \frac{\beta}{\omega_\Sigma}$$

where the above are to be thought of as quotients in the determinant lines of V^* and H^*. Let $\pi_{VV} : \Omega^2(\mathbb{P}(E)) \to \Lambda^2 V^*$ and $\pi_{HH} : \Omega^2(\mathbb{P}(E)) \to H^*$ be the projections onto the respective summands, where we are using the C^∞ splitting $\Omega^2(\mathbb{P}(E)) = \Lambda^2 V^* \oplus (V^* \otimes H^*) \oplus \Lambda^2 H^*$. Let $\alpha \in \Omega^2(\mathbb{P}(E))$.

By definition we have:

$$\begin{aligned} \Lambda_{\omega_{k,1}(t)}(\alpha) &= r\frac{\alpha \wedge (\omega_{k,1}(t))^{r-1}}{(\omega_{k,1}(t))^r} \\ &= r\frac{(\pi_{VV}(\alpha) + \pi_{HH}(\alpha)) \wedge ((\omega_{FS}(h,J_t)) + (\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma))^{r-1}}{(\omega_{FS}(h,J_t) + (\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma)^r} \\ &= (r-1)\frac{(\pi_{VV}(\alpha)) \wedge (\omega_{FS}(h,J_t))^{r-2} \wedge ((\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma)}{(\omega_{FS}(h,J_t))^{r-1} \wedge ((\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma)} \\ &\quad + \frac{\pi_{HH}(\alpha) \wedge (r\omega_{FS}(h,J_t))^{r-1}}{((\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma) \wedge (\omega_{FS}(h,J_t))^{r-1}} \\ &= (r-1)\frac{\pi_{VV}(\alpha) \wedge (r\omega_{FS}(h,J_t))^{r-2}}{(\omega_{FS}(h,J_t))^{r-1}} + \frac{\pi_{HH}(\alpha)}{((\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t}) + k)\,\omega_\Sigma)} \\ &= \Lambda_{\omega_{FS}(h,J_t)}(\pi_{VV}(\alpha)) + \frac{\pi_{HH}(\alpha)}{k\omega_\Sigma\left(\frac{\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})}{k} + 1\right)} \end{aligned} \tag{5.20}$$

$$
\begin{aligned}
&= \Lambda_{\omega_{FS(h,J_t)}}(\pi_{VV}(\alpha)) + k^{-1}\Lambda_{\omega_\Sigma}(\pi_{HH}(\alpha))\frac{1}{\left(\frac{\Phi_h^*(-\Lambda_{\omega_\Sigma}F_{A_t})}{k}+1\right)} \\
&= \Lambda_{\omega_{FS(h,J_t)}}(\pi_{VV}(\alpha)) + k^{-1}\Lambda_{\omega_\Sigma}(\pi_{HH}(\alpha)) \\
&\quad + \sum_{l=1}^{\infty}(-1)^l k^{-(l+1)}\Lambda_{\omega_\Sigma}(\pi_{HH}(\alpha))\left(\Phi_h^*(-\Lambda_{\omega_\Sigma}F_{A_t})\right)^l
\end{aligned}
$$

where in the last line we have used the Taylor expansion of $1/1+x$.

Applying this to $\rho_k(\omega_{k,1}(t))$ we therefore obtain

$$
\begin{aligned}
Scal(\omega_{k,1}(t)) &= \Lambda_{\omega_{k,1}(t)}(\rho_k(\omega_{k,1}(t))) \\
&= \Lambda_{\omega_{FS(h,J_t)}}(\pi_{\mathcal{V}\mathcal{V}}(\rho_k(\omega_{k,1}(t)))) + k^{-1}\Lambda_{\omega_\Sigma}(\pi_{HH}(\rho_k(\omega_{k,1}(t)))) \\
&\quad + \sum_{l=1}^{\infty}(-1)^l k^{-(l+1)}\Lambda_{\omega_\Sigma}(\pi_H(\rho_k(\omega_{k,1}(t)))\left(\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\right)^l \\
&= Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) \\
&\quad + k^{-1}\left(Scal(\omega_\Sigma) - r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}) + \Delta_{\mathcal{V}_t}\left(\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_t})\right) + itr(\Lambda_{\omega_\Sigma}F_{A_t})\right) \\
&\quad - \sum_{j=1}^{\infty} k^{-(j+1)}\Delta_{\mathcal{V}_t}\left((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}))^{j+1}\right) \\
&\quad + \sum_{l=1}^{\infty} k^{-(l+1)}\left(Scal(\omega_\Sigma) - r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}) + itr(\Lambda_{\omega_\Sigma}F_{A_t})\right)\left(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t})\right)^l \\
&\quad - \sum_{l=0}^{\infty}\left(\sum_{j=0}^{\infty} k^{-(j+l+2)}\Delta_{\mathcal{H}_t}\left((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}))^{j+1}\right)\right)\left(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t})\right)^l .
\end{aligned} \tag{5.21}
$$

Now we have by Lemma 4.11

$$
\begin{aligned}
& -r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}) - \Delta_{\mathcal{V}_t}\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}) + itr(\Lambda_{\omega_\Sigma}F_{A_t}) \\
= & -r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}^\circ + \frac{tr\,(\Lambda_{\omega_\Sigma}F_{A_t})}{r}Id_E) - \Delta_{\mathcal{V}_t}\left(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}^\circ + \frac{tr\,(\Lambda_{\omega_\Sigma}F_{A_t})}{r}Id_E)\right) + itr(\Lambda_{\omega_\Sigma}F_{A_t}) \\
= & -2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}^\circ) - itr\,(\Lambda_{\omega_\Sigma}F_{A_t}) + itr(\Lambda_{\omega_\Sigma}F_{A_t}) = -2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_t}^\circ),
\end{aligned}
$$

since $itr\,(\Lambda_{\omega_\Sigma}F_{A_t})$ is pulled back from Σ, and therefore annhilated by $\Delta_{\mathcal{V}_t}$. We therefore obtain 5.14. Precisely the same calculation holds for the fixed holomorphic structure J_∞ which gives the expansion for $Scal(\omega_k(h, J_\infty))$. □

The following lemma shows that if we choose our path of connections to satisfy Yang-Mills flow at the appropriate speed, as discussed at the beginning of this subsection, the resulting path of metrics actually gives a solution to Calabi flow on $\mathbb{P}(\mathcal{E})$ up to the diffeomorphisms $\tilde{g}_s$ and up to second order in powers of k^{-1}.

Lemma 5.4. *Let A_s satisfy the Yang-Mills flow at time $s = t \cdot r/k$, inducing Kähler metrics $\omega_{k,1}(s) = \omega_k(h, J_s)$ and $\widehat{\omega}_{k,1}(s) = \widetilde{g}_s^*(\omega_{k,1}(s))$. If $H(\omega_{k,1}(s))$ is as defined in Lemma 5.3, then there is an equality*

$$
rk^{-1}\left(\frac{\partial\omega_{k,1}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,1}(s)\right) = i\bar{\partial}_{J_s}\partial_{J_s}\left(\frac{2r}{k}\Phi_h(F_{A_s})\right) = i\overline{\partial}_{J_s}\partial_{J_s}H(\omega_{k,1}(s)) \tag{5.22}
$$

where V_s is the time dependent infinitesimal generator associated to $\widetilde{g}_s$. This implies

$$
\frac{\partial\widehat{\omega}_{k,1}(s)}{\partial t} = i\overline{\partial}_J\partial_J H\left(\widehat{\omega}_{k,1}(s)\right) \tag{5.23}
$$

and in particular

$$\frac{\partial \widehat{\omega}_{k,1}(s)}{\partial t} + i\bar{\partial}_J \partial_J Scal(\widehat{\omega}_{k,1}(s)) = \mathcal{O}(k^{-2}).$$

We also have

$$\begin{aligned} & i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(Scal\left(\omega_{k,1,\infty}\right) + H(\omega_{k,1,\infty})\right) \\ (5.24) \qquad = \; & i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(Scal\left(\omega_{k,1,\infty}\right) + \frac{2r}{k}\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\right) = \mathcal{O}\left(k^{-2}\right). \end{aligned}$$

Proof. Let g_s be the complex gauge transformation associated to a solution A_s of Yang-Mills flow, and V_s be the time dependent infinitesimal generator associated to $\widetilde{g}_s$, namely the one parameter family of vector fields given by

$$V_s = \frac{d(\widetilde{g_w \circ \widetilde{g}_s^{-1}})}{dw}|_{w=s}$$

of $\widetilde{g}_s^*$. By Lemma 4.9we have

$$2i\bar{\partial}_J\partial_J\left(\Phi_h(F_{h_s})\right) = \frac{\partial iF_{(h_s,\mathcal{L})}}{\partial s} = \frac{\partial\omega(h_s,J)}{\partial s} = \frac{\partial\omega_k(h_s,J)}{\partial s} = \frac{\partial\widetilde{g}_s^*(\omega_{k,1}(h,J_s))}{\partial s}.$$

We also have that

$$\frac{\partial\widetilde{g}_s^*(\omega_{k,1}(s))}{\partial s} = \widetilde{g}_s^*\left(\frac{\partial\omega_{k,1}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,1}(s)\right),$$

and therefore

$$\begin{aligned} 2rk^{-1}(i\bar{\partial}_{J_s}\partial_{J_s}(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))) &= 2rk^{-1}(\widetilde{g}_s^{-1})^*(i\bar{\partial}_J\partial_J(\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s})) \\ &= rk^{-1}(\frac{\partial\omega_{k,1}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,1}(s)), \end{aligned}$$

which is equation 5.22. In the same way, we have

$$\begin{aligned} \frac{\partial\widehat{\omega}_{k,1}(s)}{\partial t} &= rk^{-1}\frac{\partial\widehat{\omega}_{k,1}(s)}{\partial s} \\ &= \widetilde{g}_s^*\left(i\overline{\partial}_{J_s}\partial_{J_s}H\left(\omega_{k,1}(s)\right)\right) \\ &= i\overline{\partial}_J\partial_J H\left(\widehat{\omega}_{k,1}(s)\right). \end{aligned}$$

which is equation 5.23.

Recall that there is an action of each g_s on the space of Hermitian metrics given by equation , so that the family of metrics $g_s \cdot h = h_s$ solves Hermitian-Yang-Mills flow. Then applying Lemma 5.3 to the Yang-Mills flow and pulling back by the diffeomorphisms induced by g_s and using equation 4.17 and more generally, Lemma 4.12 gives

$$\begin{aligned} Scal(\widehat{\omega}_{k,1}(s)) &= Scal(\widetilde{g}_s^*(\omega_{k,1}(s)) = \widetilde{g}_s^*(Scal((\omega_{k,1}(s)) \\ &= \widetilde{g}_s^*(Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}(-2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}) + Scal(\omega_\Sigma)) \\ &+ \sum_{l=2} k^{-l}\widetilde{g}_s^*(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_h,l}(s) + \Psi_{\perp,l}(s)) \\ &= Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}(-2r\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s}) + Scal(\omega_\Sigma)) \\ &+ \sum_{l=2} k^{-l}((\Psi_{\Sigma,l}(s) + \Psi_{\Phi_{h_s},l}(s) + \Psi_{\perp,l}(s)). \end{aligned}$$

Therefore, by equation 5.23, we get

$$i\bar{\partial}_J\partial_J Scal(\hat{\omega}_{k,1}\left(s(t)\right))$$

$$= \quad -\frac{\partial \widehat{\omega}_{k,1}(s)}{\partial t} + \sum_{l=2} k^{-l}((\Psi_{\Sigma,l}(s) + \Psi_{\Phi_{h_s},l}(s) + \Psi_{\perp,l}(s)),$$

and so

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\hat{\omega}_{k,1}\left(s(t)\right)\right) + i\bar{\partial}_J\partial_J Scal(\hat{\omega}_{k,1}\left(s(t)\right)) \\ = \quad & \sum_{l=2} k^{-l}((\Psi_{\Sigma,l}(s) + \Psi_{\Phi_{h_s},l}(s) + \Psi_{\perp,l}(s)) \\ = \quad & \mathcal{O}(k^{-2}). \end{aligned}$$

The last statement is exactly the same as the last statement of Lemma 5.3. □

We now claim that each of these functions is in an appropriate parabolic Sobolev space with respect to the Kähler metric $\omega_{k,1,\infty}$ on $\mathbb{P}(\mathcal{E}_\infty)$ defined in Section 4.5. The following lemma and its corollary will be of crucial importance in the sequel.

Lemma 5.5. *For each k and each $l \geq 2$, the functions $\Psi_{\Sigma,l}(s), \Psi_{\Phi_h,l}(s), \Psi_{\perp,l}(s)$ appearing in the expansion of the scalar curvature of the Kähler metric $\omega_k(h, J_s)$, converge in $C^\infty(\mathbb{P}(E))$ to smooth functions $\Psi_{\Sigma,l,\infty}, \Psi_{\Phi_h,l,\infty}, \Psi_{\perp,l,\infty}$ on $\mathbb{P}(E)$ and for each p, q and ε, we have that the functions $\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}$, $\Psi_{\Phi_h,l,\infty}(s) - \Psi_{\Phi_h,l,\infty}$, and $\Psi_{\perp,l}(s) - \Psi_{\perp,l,\infty}$ lie in the parabolic spaces $W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})$ for all p, q, and ε and more precisely*

$$\begin{aligned} ||\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}||_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}), \\ ||\Psi_{\Phi_h,l,\infty}(s) - \Psi_{\Phi_h,l,\infty}||_{W_{4,p,q,w_\varepsilon(s)}(g_{k,\infty})} &= \mathcal{O}(k^{1/2}), \\ ||\Psi_{\perp,l}(s) - \Psi_{\perp,l,\infty}||_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})} &= \mathcal{O}(k^{1/2}). \end{aligned} \tag{5.25}$$

Proof. The precise expression for the scalar curvature equation 5.21 gives that for each $l \geq 2$

$$\begin{aligned} \Psi_l(s) = \quad & \Psi_{\Sigma,l}(s) + \Psi_{\Phi_h,l}(s) + \Psi_{\perp,l}(s) \\ = \quad & -(\Delta_{\mathcal{V}_s}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l) + \ (Scal(\omega_\Sigma) - r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}) + itr(\Lambda_{\omega_\Sigma}F_{A_t}))(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^{l-1} \\ & - \sum_{\substack{m,j \\ m+j+2=l}} \Delta_{\mathcal{H}_s}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^m. \end{aligned}$$

We have by Lemma 4.17 that $\Delta_{\mathcal{V}_s} \to \Delta_{\mathcal{V}_\infty}$ and $\Delta_{\mathcal{H}_t} \to \Delta_{\mathcal{H}_\infty}$ (in the operator norm induced by the C^p norm for each p) at a rate of $\frac{1}{\sqrt{s}}$, and by Lemma 4.15, $\Phi_h(-\Lambda_{\omega_\Sigma}F_{A_s})$ converges (and is therefore also bounded) in the C^∞ topology, at the same rate. On the other hand for the case $i = 0$, for any $p \geq 0$

$$\begin{aligned} & \left\|\Delta_{\mathcal{V}_s}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l) - \Delta_{\mathcal{V}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^l)\right\|_{C^p} \\ = \quad & \left\|(\Delta_{\mathcal{V}_s} - \Delta_{\mathcal{V}_\infty})((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l) + \Delta_{\mathcal{V}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l - (\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^l)\right\|_{C^p} \\ \leq \quad & \left\|\Delta_{\mathcal{V}_s} - \Delta_{\mathcal{V}_\infty}|| \cdot ||(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l||_{C^\beta} + ||\Delta_{\mathcal{V}_\infty}|| \cdot ||\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^l - (\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^l)\right\|_{C^p} \\ \leq \quad & C||\Delta_{\mathcal{V}_s} - \Delta_{\mathcal{V}_\infty}|| + C(||\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}) - (\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))||_{C^p} \\ & \times||(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^{l-1} + \cdots + (\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^{l-1}||_{C^p}) \\ \leq \quad & \frac{C}{\sqrt{s}}, \end{aligned}$$

for s sufficiently large, where we have used the fact that the C^p norm of a product of two functions is bounded by a constant times the product of the C^s norms, and Lemmas 4.15 and 4.17 again. Similarly we obtain

$$\begin{aligned}
& \left\| \Delta_{\mathcal{H}_s}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^m - \Delta_{\mathcal{H}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})) \right\|_{C^p} \\
= \; & ||(\Delta_{\mathcal{H}_s} - \Delta_{\mathcal{H}_\infty})((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^m||_{C^p} \\
& + ||\Delta_{\mathcal{H}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^m - \Delta_{\mathcal{H}_\infty}(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^m||_{C^p} \\
\leq \; & C||\Delta_{\mathcal{H}_s} - \Delta_{\mathcal{H}_\infty}|| + (||\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}) - \Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})||_{C^p} \\
& \times ||\Delta_{\mathcal{H}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))^{m-1} + \cdots + \Delta_{\mathcal{H}_\infty}(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^{m-1}||_{C^p}) \\
\leq \; & C||\Delta_{\mathcal{H}_s} - \Delta_{\mathcal{H}_\infty}|| + C||\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}) - \Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})||_{C^p} \\
\leq \; & \frac{C}{\sqrt{s}},
\end{aligned}$$

for s sufficiently large. Therefore setting $\Psi_{\Sigma,l,\infty}, \Psi_{\Phi_h,l,\infty}, \Psi_{\perp,l,\infty}$ equal to the the images under the projection maps (defined by the decomposition of $C^\infty(\mathbb{P}(E))$) of the function

$$\begin{aligned}
& -(\Delta_{\mathcal{V}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^l) + \; (Scal(\omega_\Sigma) - r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}) + itr(\Lambda_{\omega_\Sigma}F_{A_\infty}))((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^{l-1} \\
& - \sum_{\substack{i,j \\ i+j+2=l}} \Delta_{\mathcal{H}_\infty}((\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})^{j+1})(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))^m,
\end{aligned}$$

and applying lemma 4.13 we have $\Psi_{\Sigma,l}(s), \Psi_{\Phi_h,l}(s), \Psi_{\perp,l}(s) \to \Psi_{\Sigma,l,\infty}, \Psi_{\Phi_h,l,\infty}, \Psi_{\perp,l,\infty}$ in C^∞ at a rate of $\frac{1}{\sqrt{s}}$.

For any $i \geq 1$

$$\left\| \partial_s^i(\Delta_{\mathcal{V}_s}) \right\|_{C^p}, \left\| \partial_s^i(\Delta_{\mathcal{H}_t}) \right\|_{C^p}, \left\| \partial_s^i(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})) \right\|_{C^p} = \left\| \Phi_h(\partial_s^i \Lambda_{\omega_\Sigma}F_{A_s}) \right\|_{C^p} \leq \frac{C}{\sqrt{s}}$$

for all p and and all s sufficiently large, where we again use Lemmas 4.15 and 4.17, and the fact that Φ_h is independent of t, and where $\left\|\partial_s^i(\Delta_{\mathcal{V}_s})\right\|_{C^p}, \left\|\partial_s^i(\Delta_{\mathcal{V}_s})\right\|_{C^p}$ are the operator norms induced by the C^β norms. All these quantities are in particular bounded, so that for $i \geq 1$ we have

$$\left\| \partial_s^i(\Psi_l(s)) \right\|_{C^p} \leq C \left(\sum_{\alpha=1}^{i} \|\partial_t^\alpha(\Delta_{\mathcal{V}_s})\|_{C^p} + \|\partial_s^\alpha(\Delta_{\mathcal{H}_s})\|_{C^p} + \|\Phi_h(\partial_s^\alpha \Lambda_{\omega_\Sigma}F_{A_s})\|_{C^p} \right) \leq \frac{C}{\sqrt{s}},$$

and in particular $\partial_s^i(\Psi_l(s)) \to 0$ in C^∞.

Now comparing the C^p norms with the Sobolev norms, we have in particular that for every $i \geq 1$ and $s \geq 0$

$$\begin{aligned}
\left\| \partial_s^i(\Psi_l(s)) \right\|_{L^2_p(g_{k,1,\infty})} & \leq (vol(\mathbb{P}(E), g_{k,1,\infty}))^{1/2} \left\| \partial_s^i(\Psi_l(s)) \right\|_{C^p(g_{k,1,\infty})} \\
\|\Psi_l(s) - \Psi_{l,\infty}\|_{L^2_p(g_{k,1,\infty})} & \leq (vol(\mathbb{P}(E), g_{k,1,\infty}))^{1/2} \|\Psi_l(s) - \Psi_{l,\infty}\|_{C^p(g_{k,1,\infty})}.
\end{aligned}$$

Now we can argue just as in [F], using Theorem 5.2 of that reference to prove that the volume form on a sufficiently small ball B around any point $p \in \Sigma$ is equal to $\mathcal{O}(k)$ times a fixed form, and therefore $vol(B, g_{k,1,\infty}) = \mathcal{O}(k)$. Covering Σ by balls of this type and summing up the volumes therefore gives $vol(\mathbb{P}(E), g_{k,1,\infty}) = \mathcal{O}(k)$. Notice also that

$$\begin{aligned}
\pi_{\Sigma *}(\partial_s^i(\Psi_l(s)) &= \partial_s^i(\pi_{\Sigma *}\Psi_l(s)) = \partial_s^i \Psi_{\Sigma,l}(s) \\
\pi_{\Phi_h *}(\partial_s^i(\Psi_l(s)) &= \partial_s^i(\pi_{\Phi_h *}\Psi_l(s)) = \partial_s^i \Psi_{\Phi_h,l}(s) \\
\pi_{\perp *}(\partial_s^i(\Psi_l(s)) &= \partial_s^i(\pi_{\pi_{\perp *}}\Psi_l(s)) = \partial_s^i \Psi_{\perp,l}(s),
\end{aligned}$$

since ∂_s^i commutes with pullback and with Φ_h. Therefore, by Lemma 4.13 we obtain that for each $i \geq 1$

$$\begin{aligned} \left\|\partial_s^i \Psi_{\Sigma,l}(s)\right\|_{L^2_p(g_{k,1,\infty})}, \left\|\partial_s^i \Psi_{\Phi_h,l}(s)\right\|_{L^2_p(g_{k,1,\infty})}, \left\|\partial_s^i \Psi_{\pi_{\perp *},l}(s)\right\|_{L^2_p(g_{k,1,\infty})} &\leq \frac{\mathcal{O}(k^{1/2})}{s} \\ \|\Psi_{\Sigma,l} - \Psi_{\Sigma,l,\infty}\|_{L^2_p(g_{k,1,\infty})}, \|\Psi_{\Phi_h,l} - \Psi_{\Phi_h,l,\infty}\|_{L^2_p(g_{k,1,\infty})}, \|\Psi_{\perp,l}(s) - \Psi_{\perp,l,\infty}\|_{L^2_p(g_{k,1,\infty})}, &\leq \frac{\mathcal{O}(k^{1/2})}{s}. \end{aligned}$$

We then obtain for the parabolic Sobolev norms

$$\begin{aligned} \|\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}\|^2_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \sum_{i=0}^{q} \int_0^\infty |w_\varepsilon(s)|^2 \left\|\partial_t^i \left(\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}\right)\right\|^2_{L^2_{4(p-i)}(g_{k,1,\infty})} \\ &= \int_0^\infty |w_\varepsilon(s)|^2 \|\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}\|^2_{L^2_{4p}(g_{k,1,\infty})} \\ &\quad + \sum_{i=1}^{q} \int_0^\infty |w_\varepsilon(s)|^2 \left\|\partial_t^i \Psi_{\Sigma,l}(s)\right\|^2_{L^2_{4(p-i)}(g_{k,1,\infty})} \\ &\leq \mathcal{O}(k) \int_0^\infty \frac{|w_\varepsilon(s)|^2}{s} = \mathcal{O}(k), \end{aligned}$$

by the definition of the weight function. The other parabolic norms are computed in exactly the same way. □

Combining the previous two lemmas, we obtain the crucial fact that our ansatz is close to a solution of (after pulling back by the diffeomorphism induced by g_s) to Calabi flow with respect to the parabolic Sobolev norms.

Corollary 5.6. *Let A_s satisfy the Yang-Mills flow at time $s = 2r/k \cdot t$, and $\omega_{k,1}(s) = \omega_k(h, J_s)$ be the resulting family of Kähler forms on $\mathbb{P}(\mathcal{E}_s)$, so that $\widehat{\omega}_{k,1}(s) = \widetilde{g}_s^*(\omega_{k,1}(s))$ is a family of Kähler forms on the fixed complex manifold $\mathbb{P}(\mathcal{E})$. Then for all p,q, and ε there is an estimate:*

$$\left\|Scal(\omega_{k,1}(s)) + \frac{2r}{k}\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_s}) - (Scal(\omega_{k,1,\infty}) + (\frac{2r}{k}\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_\infty}))\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} \leq Ck^{-3/2}.$$

Proof. By Lemma 5.4 we have a pointwise expansion

$$\begin{aligned} & Scal(\omega_{k,l}(s)) + \frac{2r}{k}\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_s}) - (Scal(\omega_{k,l,\infty}) + \frac{2r}{k}\Phi_h(\Lambda_{\omega_\Sigma} F^\circ_{A_\infty})) \\ = & \sum_{l=2} k^{-l}((\Psi_{\Sigma,l}(t) - \Psi_{\Sigma,l,\infty}) + (\Psi_{\Phi_h,l}(t) - \Psi_{\Phi_h,l,\infty}) + (\Psi_{\perp,l}(t) - \Psi_{\perp,l,\infty}). \end{aligned}$$

By the previous Lemma we obtain the result.

□

Applying Lemmas 5.4 and 5.5 and Corollary 5.6 gives Theorem 5.1 for $l = 1$.

5.3. **The Second order correction.** In this subsection we will prove Theorem 5.1 in the case $l = 2$. This is the main step in the induction. More specifically we will prove the following proposition.

Proposition 5.7. *Fix any $S \in [0, \infty]$. There is a path $\eta_s^S \in \mathfrak{su}(E)$, and one parameter families of Kähler potentials $\Theta(s(t)) \in \pi^* C^\infty(\Sigma)$, $\Xi(\eta_s^S) \in C^\infty(\mathbb{P}(E))$, $\Omega(s(t)) \in C^\infty(\mathbb{P}(E))_\perp$ converging smoothly to functions Θ_∞,Ξ_∞, Ω_∞ so that the path of Kähler forms*

$$\omega_{k,2}(s) = \omega_{k,1}(s) + i\overline{\partial}_{J_s}\partial_{J_s}\Theta(s(t)) + k^{-1} i\overline{\partial}_{J_s}\partial_{J_s}\Xi((s(t)) + k^{-2} i\overline{\partial}_{J_s}\partial_{J_s}\Omega(s(t)) \tag{5.26}$$

compatible with the holomorphic structure J_s converges to a form $\omega_{k,2,\infty}$ (with corresponding metric $g_{k,2,\infty}$) compatible with J_∞.

Moreover, there exists a one parameter family of functions $H(\omega_{k,2}(s))$, converging to a function $H(\omega_{k,2,\infty})$ such that the following properties hold.

Pointwise there is an equation:

$$\begin{aligned} &Scal(\omega_{k,2}(s)) + H(\omega_{k,2}(s)) - (Scal(\omega_{k,2,\infty}) + H(\omega_{k,2,\infty})) \\ = &\sum_{l=3} k^{-l}(\Psi^{(2)}_{\Sigma,l}(s) - \Psi^{(2)}_{\Sigma,l,\infty}) + (\Psi^{(2)}_{\Phi_h,l}(s) - \Psi^{(2)}_{\Phi_h,l,\infty}) + (\Psi^{(2)}_{\perp,l}(s) - \Psi^{(2)}_{\perp,l,\infty})), \end{aligned} \tag{5.27}$$

where $\Psi^{(2)}_{\Sigma,l}(s), \Psi^{(2)}_{\Phi_h,l}(s)$, and $\Psi^{(2)}_{\perp,l}(s)$ are smooth families of functions (each belonging to the respective summand of $C^\infty(\mathbb{P}(E))$) converging in $C^\infty(\mathbb{P}(E))$ to the smooth functions $\Psi^{(2)}_{\Sigma,l,\infty}, \Psi^{(2)}_{\Phi_h,l,\infty}, \Psi^{(2)}_{\perp,l,\infty}$.

We furthermore have an equality

$$rk^{-1}\left(\frac{\partial \omega_{k,2}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,2}(s)\right) = i\bar{\partial}_{J_s}\partial_{J_s}H(\omega_{k,2}(s)), \tag{5.28}$$

which implies in particular that

$$\frac{\partial \widehat{\omega}_{k,2}(s)}{\partial t} = i\bar{\partial}_J\partial_J H(\widehat{\omega}_{k,2}(s)),$$

for all $s \in [0,S]$.

Equivalently we obtain a formal solution

$$\frac{\partial \widehat{\omega}_{k,2}(s)}{\partial t} + i\overline{\partial}_J\partial_J Scal(\widehat{\omega}_{k,2}(s)) = \mathcal{O}(k^{-3}) \tag{5.29}$$

to Calabi flow on $\mathbb{P}(\mathcal{E})$ to order 3 in k^{-1}, for all $s \in [0,S]$.

Finally, for all p, q and ε we have

$$\begin{aligned} ||\Psi^{(2)}_{\Sigma,l}(s) - \Psi^{(2)}_{\Sigma,l,\infty}||^2_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}), \\ ||\Psi^{(2)}_{\Phi_h,l}(s) - \Psi^{(2)}_{\Phi_h,l,\infty}||^2_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}), \\ ||\Psi^{(2)}_{\perp,l}(s) - \Psi^{(2)}_{\perp,l,\infty}||^2_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}). \end{aligned} \tag{5.30}$$

which implies an estimate

$$\|Scal(\omega_{k,2}(s)) + H(\omega_{k,2}(s)) - (Scal(\omega_{k,2,\infty}) + H(\omega_{k,2,\infty}))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} \leq Ck^{-(5/2)}. \tag{5.31}$$

The remainder of this subsection will consist of the proof of 5.7. From the previous subsection we may write

$$\frac{\partial \widehat{\omega}_{k,1}(s)}{\partial t} + i\overline{\partial}_J\partial_J Scal(\widehat{\omega}_{k,1}(s)) = k^{-2}\widetilde{g}_s^*(\Psi_{\Sigma,2}(s) + \Psi_{\Phi_h,2}(s) + \Psi_{\perp,2}(s)) + \mathcal{O}(k^{-3}).$$

The goal is to add Kähler potentials to $\widehat{\omega}_{k,1}(s)$ in order to eliminate the first three terms. We will handle the terms involving $\Psi_{\Sigma,2}(s)$, $\Psi_{\Phi_h,2}(s)$, and $\Psi_{\perp,2}(s)$, in that order, by adding three new potentials, one for each summand. Each time we add a potential, we first calculate the effect of the change in the metric this induces on the scalar curvature, and see that in order to eliminate the relevant term of order 2 in k^{-1}, we must solve a linear parabolic equation of the type discussed in the appendix. The key point is that when we add each potential, we will only change the right hand side of the equation above at orders 3 and above in k^{-1} by terms involving the added potentials. The parabolic theory, together with the estimates obtained in the last subsection will then allow us to obtain estimates on the potentials, which will in turn give us estimates on the $\mathcal{O}(k^{-3})$ terms as well, as in the statement of Proposition 5.7.

Proof. **Step 1: Correcting $\Psi_{\Sigma,2}$.**

We will start by eliminating $\Psi_{\Sigma,2}(s(t))$. To do so we will modify the metric ω_Σ on Σ. Since $\omega_{k,1}(t)=\omega(h,J_s)+k\omega_\Sigma$, modifying ω_Σ by adding $k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w(t))$ for some one parameter family of functions $\Theta(w(t))\in C^\infty(\Sigma)$ is the same as modifying $\omega_{k,1}(t)$ by adding $\pi^*(i\bar\partial_\Sigma\partial_\Sigma\Theta(w(t)))$. Here $w(t)=\frac{t}{k^2}$.

So we obtain a new metric

$$\begin{aligned}\omega'_{k,1}(t) &= \omega'_{k,1}(s(t),w(t))\\ &= \omega_{k,1}(s(t))+\pi^*(i\bar\partial_\Sigma\partial_\Sigma\Theta(w(t))),\end{aligned}\tag{5.32}$$

and to calculate the effect of this change on the scalar curvature we simply replace ω_Σ by $\omega'_\Sigma=\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w(t)))$, in the expression 5.21 obtained in the proof of Lemma 5.3 that is:

$$\begin{aligned}Scal(\omega'_{k,1}(s(t))) &= Scal(\omega_{k,1}(s(t))+i\bar\partial_\Sigma\partial_\Sigma\Theta(w)) = Scal(\omega(h,J_s)+k(\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w))\\ &= Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right)+k^{-1}\left(-2r\Phi_h(\Lambda_{\omega_\Sigma+i\bar\partial_{J_s}\partial_{J_s}\Theta(w)}F^\circ_{A_t})+Scal(\omega_\Sigma+i\bar\partial_{J_s}\partial_{J_s}\Theta(w))\right)\\ &\quad+\sum_{l=2}k^{-2}\left(\Psi_{\Sigma,l}(s)+\Psi_{\Phi_h,l}(s)+\Psi_{\perp,l}(s))\right)+\mathcal{O}(k^{-3}).\end{aligned}\tag{5.33}$$

Now we compute all the expressions in the above formula, beginning with $Scal(\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma)$. We have

$$\begin{aligned}Scal(\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)) &= \Lambda_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}(\rho_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)})\\ &= \frac{\rho_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}}{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}=\frac{\Lambda_{\omega_\Sigma}(\rho_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)})}{(1+k^{-1}\Delta_\Sigma\Theta(w))}.\end{aligned}$$

Below we will define $\Theta(w)$ as the solution of a parabolic equation, and in particular it will be bounded as $w\to\infty$, so that for k sufficiently large we have $|k^{-1}\Delta_\Sigma\Theta(w)|<1$. Therefore we have a pointwise expansion of the form

$$Scal(\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w))=\Lambda_{\omega_\Sigma}(\rho_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)})\left(\sum_{i=0}^\infty(-1)^ik^{-i}\left(\Delta_{\omega_\Sigma}\Theta(w)\right)^i\right).$$

We also have

$$\begin{aligned}\rho_{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}-\rho_{\omega_\Sigma} &= i\bar\partial_\Sigma\partial_\Sigma\log\left(\frac{\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}{\omega_\Sigma}\right)\\ &= i\bar\partial_\Sigma\partial_\Sigma\log(1+k^{-1}\Delta_{\omega_\Sigma}\Theta(w))\\ &= i\bar\partial_\Sigma\partial_\Sigma\left(\sum_{i=1}^\infty(-1)^{i+1}k^{-i}\frac{(\Delta_{\omega_\Sigma}\Theta(w))^i}{i}\right)\end{aligned}$$

Then we obtain

$$\begin{aligned}Scal(\omega_\Sigma+k^{-1}i\bar\partial_\Sigma\partial_\Sigma\Theta(w)) &= \Lambda_{\omega_\Sigma+i\bar\partial_\Sigma\partial_\Sigma\Theta(w)}(\rho_{\omega_\Sigma+i\bar\partial_\Sigma\partial_\Sigma\Theta(w)})\\ &= \Lambda_{\omega_\Sigma}\left(\rho_{\omega_\Sigma}+i\bar\partial_\Sigma\partial_\Sigma\left(\sum_{i=1}^\infty(-1)^{i+1}k^{-i}\frac{(\Delta_{\omega_\Sigma}\Theta(w))^i}{i}\right)\right)\sum_{i=0}^\infty(-1)^ik^{-i}\left(\Delta_{\omega_\Sigma}\Theta(w)\right)^i\\ &= Scal(\omega_\Sigma)+k^{-1}\left(\Delta^2_{\omega_\Sigma}\Theta(w)-Scal(\omega_\Sigma)\Delta_{\omega_\Sigma}\Theta(w)\right)+\mathcal{O}(k^{-2})\\ &= Scal(\omega_\Sigma)+k^{-1}\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)+\mathcal{O}(k^{-2}),\end{aligned}$$

Where we have used Lemma 2.5, to conclude that

$$\left(\Delta^2_{\omega_\Sigma}\Theta(w) - Scal(\omega_\Sigma)\Delta_{\omega_\Sigma}\Theta(w)\right) = (dScal_{\omega_\Sigma})_0(\Theta(w)) = \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w).$$

since $Scal(\omega_\Sigma)$ is constant. For the moment the above expression is purely formal, but we will make it precise in the sequel.

We may also compute

$$\begin{aligned}
\Lambda_{\omega_\Sigma + k^{-1}i\bar{\partial}_\Sigma\partial_\Sigma\Theta(w)}F^\circ_{A_s} &= \frac{F^\circ_{A_s}}{\omega_\Sigma + k^{-1}i\bar{\partial}_\Sigma\partial_\Sigma\Theta(w)} = \Lambda_{\omega_\Sigma}F^\circ_{A_s}\left(\frac{1}{1+k^{-1}\left(\frac{i\bar{\partial}_\Sigma\partial_\Sigma\Theta(w)}{\omega_\Sigma}\right)}\right) \\
&= \Lambda_{\omega_\Sigma}F^\circ_{A_s} - k^{-1}\Lambda_{\omega_\Sigma}F^\circ_{A_s}\left(\Delta_{\omega_\Sigma}\Theta(w)\right) + \sum_{i=2}^{\infty}k^{-i}\Lambda_{\omega_\Sigma}F^\circ_{A_s}((\Delta_{\omega_\Sigma}\Theta(w))^i).
\end{aligned}$$

Expanding the expression 5.33, we obtain

$$\begin{aligned}
& Scal(\omega'_{k,1}(s(t))) \\
= \; & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}\left(Scal(\omega_\Sigma) - 2r(\Phi_h(\Lambda_{\omega_\Sigma}F^\circ_{A_s}))\right) \\
& + k^{-2}\left((\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w) + 2r(\Phi_h(\Lambda_{\omega_\Sigma}F^\circ_{A_s})\Delta_{\omega_\Sigma}\Theta(w)\right) \\
& + \sum_{l=2}k^{-l}(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_h,l}(s) + \Psi_{\perp,l}(s)).
\end{aligned}$$

Remark 5.8. The two key points here are (1) that we have not changed the k^{-1} term at all, so that the new metric will still give an approximation to Calabi flow at order 1, and only slighly modified the k^{-2} term by the expression

$$\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)) + 2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_h(\Lambda_{(\omega_\Sigma}F^\circ_{A_s}),$$

which will help us kill $\Psi_{\Sigma,2}(t)$, and it is otherwise unchanged; and (2) the new $\mathcal{O}(k^{-3})$ term will remain in the appropriate parabolic Sobolev space as we shall see below. For the time being, to lighten the notation, we continue to denote these latter terms by $\Psi_{\Sigma,l}(s), \Psi_{\Phi_h,l}(s), \Psi_{\perp,l}(s))$, even though stricitly speaking they have been modified. We will modify the notation later, after we have we have constructed all of the required potentials.

Now we define $\Theta(w)$. There is a solution to the elliptic equation

$$\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta_\infty = -\widehat{\Psi}_{\Sigma,2,\infty}, \tag{5.34}$$

(where we will abuse notation here and leave out the pullback symbol, and where $\widehat{\Psi}_{\Sigma,2,\infty}$ denotes the difference with the mean value), since by definition

$$\int_\Sigma \widehat{\Psi}_{\Sigma,2,\infty} = 0,$$

and therefore

$$-\widehat{\Psi}_{\Sigma,2,\infty} \perp \ker \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma},$$

since

$$\ker \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma} = \mathbb{R}$$

We take $\widetilde{\Theta}(w)$ to be the solution of the linear parabolic initial value problem

$$\frac{\partial\widetilde{\Theta}(w)}{\partial w} + \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\widetilde{\Theta}(w) = -(\widehat{\Psi}_{\Sigma,2}(s) - \widehat{\Psi}_{\Sigma,2,\infty}), \tag{5.35}$$

$$\widetilde{\Theta}(0) = -\Theta_\infty,$$

the longtime existence of which is provided by Theorem 7.10, which we may apply using the facts that $\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}$ is a semi-defiinite self adjoint operator whose kernel (which again is $\mathbb{C}$) , is orthogonal to $\widehat{\Psi}_{\Sigma,2}(s) - \widehat{\Psi}_{\Sigma,2,\infty}$ for all s (and so all w) and that by Lemma 5.5 we also have

$$\widehat{\Psi}_{\Sigma,2}(s) - \widehat{\Psi}_{\Sigma,2,\infty} \in W_{4,p,q,w_{\varepsilon(s)}}(g_{k.1.\infty}).$$

Now we define

$$\Theta(w) = \widetilde{\Theta}(w) + \Theta_\infty, \tag{5.36}$$

which then satisfies the initial value equation

$$\frac{\partial \Theta(w)}{\partial w} + \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w) = -\widehat{\Psi}_{\Sigma,2}(s). \tag{5.37}$$
$$\Theta(0) = 0$$

We remark here that by the regularity theory for parabolic equations

$$\widetilde{\Theta}(w) = \Theta(w) - \Theta_\infty$$

is also in the parabolic space $W_{4,p+1,q,w_\varepsilon(s)}(g_{k.1.\infty})$ and satisfies the parabolic estimate

$$\begin{aligned}
& ||\Theta(w) - \Theta_\infty||_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k.1.\infty})} \\
\leq\ & C\left(\|\Theta_\infty\|_{L^2_{4p+2}(g_{k.1.\infty})} + \left\|-(\widehat{\Psi}_{\Sigma,2}(s) - \widehat{\Psi}_{\Sigma,2,\infty})\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k.1.\infty})}\right) \\
\leq\ & Ck^{1/2},
\end{aligned} \tag{5.38}$$

again by Lemma 5.5.

We also define

$$H(\omega'_{k,1}(s(t)) = k^{-1}(2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})) - k^{-2}\left(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w) + \widehat{\Psi}_{\Sigma,2}(s)\right). \tag{5.39}$$

Then note that one has the analogue of equation 5.28, namely:

$$\begin{aligned}
& i\overline{\partial}_J\partial_J\left(H(\omega'_{k,1}(s(t))\right) \\
=\ & k^{-1}(2ri\overline{\partial}_J\partial_J\left(\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})\right)) - k^{-2}i\overline{\partial}_J\partial_J\left(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w) + \widehat{\Psi}_{\Sigma,2}(s)\right) \\
=\ & 2rk^{-1}(\frac{\partial\omega_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}(\omega_{k,1}(s(t))) + k^{-2}\frac{\partial}{\partial w}\left(i\overline{\partial}_\Sigma\partial_\Sigma\Theta(w)\right) \\
=\ & 2rk^{-1}\left(\frac{\partial\omega_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}(\omega_{k,1}(s(t)) + \frac{\partial}{\partial s}\left(i\overline{\partial}_\Sigma\partial_\Sigma\Theta(w)\right)\right) \\
=\ & 2rk^{-1}\left(\frac{\partial\omega'_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}(\omega'_{k,1}(s(t))\right),
\end{aligned} \tag{5.40}$$

where we have used the fact that

$$\mathcal{L}_{V_s}(i\overline{\partial}_\Sigma\partial_\Sigma\Theta(w)) = i\overline{\partial}_\Sigma\partial_\Sigma\left(\mathcal{L}_{V_s}\left(\Theta(w)\right)\right) = 0,$$

because

$$\begin{aligned}
\mathcal{L}_{V_s}\left(\Theta(w)\right) &= V_s\left(\Theta(w)\right) = \frac{d}{d\varsigma}(\widetilde{g}_\varsigma \circ \widetilde{g}_s^{-1}|_{\varsigma=s})\left(\Theta(w)\right) \\
&= \frac{d}{d\varsigma}(\Theta(w) \circ \widetilde{g}_\varsigma \circ \widetilde{g}_s^{-1}|_{\varsigma=s}) = 0,
\end{aligned}$$

since $\Theta(w) \circ \widetilde{g}_\varsigma \circ \widetilde{g}_s^{-1}$ is constant, because $\Theta(w)$ is constant on the fibres of $\mathbb{P}(E)$ and $\widetilde{g}_\varsigma \circ \widetilde{g}_s^{-1}$ preserves the fibres by definition.

Then formally we obtain

$$
\begin{aligned}
& Scal(\omega'_{k,1}(s(t))) + H(\omega'_{k,1}(s(t)) \\
= \;& Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\,(\omega_\Sigma) \\
+ \;& k^{-2}(2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_h(\Lambda_{\omega_\Sigma}F^\circ_{A_s}) + \Psi_{\Phi_h,2}(t) + \Psi_{\perp,2}(t)) \\
& + \sum_{l=3} k^{-l}(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_h,l}(s) + \Psi_{\perp,l}(s)),
\end{aligned} \tag{5.41}
$$

which by equation 5.40 gives an initial version of equation 5.29, namely the equation:

$$
\begin{aligned}
& \frac{\partial \widehat{\omega}'_{k,1}(s(t))}{\partial t} + i\overline{\partial}_J\partial_J Scal(\widehat{\omega}'_{k,1}(s(t))) \\
= \;& \frac{\partial \widetilde{g}^*_s(\omega'_{k,1}(s(t)))}{\partial t} + i\overline{\partial}_J\partial_J Scal(\widetilde{g}^*_s(\omega'_{k,1}(s(t)))) \\
= \;& 2rk^{-1}\widetilde{g}^*_s\left(\frac{\partial \omega'_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}(\omega'_{k,1}(s(t))\right) + \widetilde{g}^*_s\left(i\overline{\partial}_{J_s}\partial_{J_s}Scal(\omega'_{k,1}(s(t)))\right) \\
= \;& \widetilde{g}^*_s\left(i\overline{\partial}_{J_s}\partial_{J_s}\left(H(\omega'_{k,1}(s(t)) + Scal(\omega'_{k,1}(s(t))\right)\right) \\
= \;& k^{-2}i\overline{\partial}_J\partial_J\left((2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_{h_s}(\Lambda_{(\omega_\Sigma}F^\circ_{h_s}) + \Psi_{\Phi_{h_s},2}(s) + \Psi_{\perp,2}(s))\right) + \mathcal{O}(k^{-3}).
\end{aligned}
$$

Notice that taking Θ_∞ as above, and defining

$$
\omega'_{k,1,\infty} = \omega_{k,1,\infty} + i\overline{\partial}_\Sigma\partial_\Sigma\Theta_\infty,
$$

and

$$
\begin{aligned}
H(\omega'_{k,1,\infty}) \;:&= H\left(\omega_{k,1,\infty}\right) \\
&= 2rk^{-1}\left(\Phi_h\left(\Lambda_{\omega_\Sigma}F^\circ_{A_\infty}\right)\right),
\end{aligned} \tag{5.42}
$$

then by the elliptic analogue of exactly the same argument above, we have an expansion

$$
\begin{aligned}
& Scal\left(\omega'_{k,1,\infty}\right) + H\left(\omega'_{k,1,\infty}\right) \\
= \;& Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\,(\omega_\Sigma) \\
+ \;& k^{-2}(2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_h(\Lambda_{\omega_\Sigma}F^\circ_{A_\infty}) + \Psi_{\Phi_h,2,\infty} + \Psi_{\perp,2,\infty}) \\
& + \sum_{l=3} k^{-l}(\Psi_{\Sigma,l,\infty} + \Psi_{\Phi_h,l,\infty} + \Psi_{\perp,l,\infty}),
\end{aligned} \tag{5.43}
$$

and subtracting equation5.41 from equation 5.43 gives a preliminary version of equation 5.27, namely:

$$
\begin{aligned}
& Scal\left(\omega'_{k,1}(s(t))\right) + H(\omega'_{k,1}(s(t)) - \left(Scal\left(\omega'_{k,1,\infty}\right) + H(\omega'_{k,1,\infty})\right) \\
= \;& k^{-2}\left(2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_s}\right) - 2r\Delta_{\omega_\Sigma}\Theta_\infty\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\right) \\
& + k^{-2}\left((\Psi_{\Phi_h,l}(s) - \Psi_{\Phi_h,l,\infty}) + (\Psi_{\perp,l}(s) - \Psi_{\perp,l,\infty})\right) \\
& + \sum_{l=3} k^{-l}\left((\Psi_{\Sigma,l}(s) - \Psi_{\Sigma,l,\infty}) + (\Psi_{\Phi_h,l}(s) - \Psi_{\Phi_h,l,\infty}) + (\Psi_{\perp,l}(s) - \Psi_{\perp,l,\infty})\right)
\end{aligned} \tag{5.44}
$$

Step 2: Correcting $\Psi_{\Phi_h,2}(s)$

We will now eliminate the term $\Psi_{\Phi_h,2}(s)$. This will be done altering the metric h on E. Namely we will define the metric

$$h_{\eta(s)} = h + k^{-1}h \cdot \eta(s) \tag{5.45}$$

where $\eta(s) \in i\mathfrak{u}(E,h)$ is a 1-parameter family of h self-adjoint endomorphisms of E. Of course the gauge transformations g_s^{-1} act on $h_{\eta(s)}$ to give

$$\begin{aligned} h_{\hat{\eta}(s)} &= g_s^{-1} \cdot h_{\eta(s)} = h_s + k^{-1} g_s^{-1} \cdot (h \cdot \eta(s)) \\ &= h_s + k^{-1} h_s \cdot \hat{\eta}_s, \end{aligned} \tag{5.46}$$

where $\hat{\eta}(s) = g_s^{-1} \circ \eta(s) \circ g_s$.

We define

$$\begin{aligned} \omega''_{k,1}(t) &= \omega''_{k,1}(s(t), w(t)) = \omega(h_{\eta(s)}, J_s) + k\omega_\Sigma + i\bar{\partial}_\Sigma \partial_\Sigma \Theta(w) \\ \hat{\omega}''_{k,1}(t) &= g_s^* \left(\omega''_{k,1}(s(t), w(t)) \right) = \omega(h_{\hat{\eta}(s)}, J) + k\omega_\Sigma + i\bar{\partial}_\Sigma \partial_\Sigma \left(\Theta(w) \right). \end{aligned}$$

Notice that we may also write:

$$\begin{aligned} \omega''_{k,1}(s(t))([v]) &= \omega'_{k,1}(s(t))([v]) + i\overline{\partial}_{J_s}\partial_{J_s} \log\left(\frac{h_{\eta_s}(v,v)}{h(v,v)}\right) \\ &= \omega'_{k,1}(s(t))([v]) + i\overline{\partial}_{J_s}\partial_{J_s} \log\left(1 + k^{-1}\frac{h(\eta_s v, v)}{h(v,v)}\right) \\ &= \omega'_{k,1}(s(t))([v]) + i\overline{\partial}_{J_s}\partial_{J_s} \log(1 + k^{-1}\Phi_h\left(-i\eta_s([v])\right)), \end{aligned}$$

so that

$$\begin{aligned} \omega''_{k,1}(s(t)) &= \omega'_{k,1}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s} \log(1 + k^{-1}\Phi_h\left(-i\eta_s\right)) \\ &= \omega'_{k,1}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s} \left(k^{-1} \sum_{i=1}^{\infty} (-1)^{i+1} k^{-(i-1)} \left(\Phi_h\left(-i\eta_s\right)\right)^i \right) \\ &= \omega'_{k,1}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s} \left(-k^{-1} \sum_{i=1}^{\infty} (k^{-(i-1)} \left(\Phi_h\left(i\eta_s\right)\right)^i \right) \\ & \quad\; \omega'_{k,1}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s} \left(k^{-1}\Xi(\eta_s) \right), \end{aligned}$$

where we define

$$\Xi(\eta_s) = -\sum_{i=1}^{\infty} (k^{-(i-1)} \left(\Phi_h\left(i\eta_s\right)\right)^i. \tag{5.47}$$

Therefore changing the metric to h_{η_s} is equivalent to adding the two form defined by the potential $\Xi(\eta_s)$.

To see how to eliminate the term $\Psi_{\Phi_h,2}$, we will begin by calculating expressions for

$$\frac{\partial \widehat{\omega}''_{k,1}(s(t))}{\partial t} \text{ and } Scal\left(\widehat{\omega}''_{k,1}(s(t))\right),$$

because the calculations are more straightforward in the framework of the metrics $\widehat{\omega}''_{k,1}(s(t))$ rather than $\omega''_{k,1}(s(t))$.

By Lemma 4.9 we have that

$$\frac{\partial \widehat{\omega}''_{k,1}(s(t))}{\partial t} = rk^{-1} \left(\frac{\partial \widehat{\omega}''_{k,1}(s(t))}{\partial s} \right) = rk^{-1} \left(\frac{\partial \omega(h_{\widehat{\eta_s}}, J)}{\partial s} + i\partial_J\partial_J\Theta(w) \right)$$

$$
\begin{aligned}
(5.48) \qquad &= rk^{-1}\left(\frac{\partial\omega(h_{\widehat{\eta_s}},J)}{\partial s}+\frac{\partial\widehat{\omega}'_{k,1}(s(t))}{\partial s}-\frac{\partial\omega(h_s,J)}{\partial s}\right)\\
&= \frac{\partial\widehat{\omega}'_{k,1}(s(t))}{\partial t}+rk^{-1}\left(\frac{\partial\omega(h_{\widehat{\eta_s}},J)}{\partial s}-\frac{\partial\omega(h_s,J)}{\partial s}\right)\\
&= \frac{\partial\widehat{\omega}'_{k,1}(s(t))}{\partial t}+rk^{-1}i\partial_J\partial_J\left(\Phi_{h_{\widehat{\eta_s}}}\left(ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}\right)-\Phi_{h_s}\left(ih_s^{-1}\frac{\partial h_s}{\partial s}\right)\right).
\end{aligned}
$$

Similarly, according to equation 5.34 after replacing h by h_{η_s} and pulling back by $\widetilde{g}_s$, we have

$$
\begin{aligned}
Scal\left(\widehat{\omega}''_{k,1}(s(t))\right) &= Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right)+k^{-1}\left(\left(Scal(\omega_\Sigma)-2r(\Phi_{h_{\widehat{\eta_s}}}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta_s}}})\right)\right)\\
&\quad +k^{-2}(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)+2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_{h_{\widehat{\eta_s}}}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta_s}}})+\Psi_{\Sigma,2}(t)+\Psi_{\Phi_{h_{\widehat{\eta_s}}},2}(t)+\Psi_{\perp,2}(t))\\
&\quad +\sum_{l=3}k^{-l}(\Psi_{\Sigma,l}(s)+\Psi_{\Phi_{h_{\widehat{\eta_s}}},l}(s)+\Psi_{\perp,l}(s)).
\end{aligned}
$$

In order to obtain a more precise expression for each of these functions in terms of $\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s})$, we must calculate the quantities

$$
\Phi_{h_{\widehat{\eta_s}}}\left(ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}\right)\ \text{and}\ \Phi_{h_{\widehat{\eta_s}}}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta_s}}}).
$$

We have by definition of $ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}$

$$
\begin{aligned}
\Phi_{h_{\widehat{\eta_s}}}\left(ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}\right)([v]) &= \sqrt{-1}\frac{h_{\widehat{\eta_s}}(ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}v,v)}{h_{\widehat{\eta_s}}(v,v)}=\sqrt{-1}\frac{ih_{\widehat{\eta_s}}(v,h_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}v)}{h_{\widehat{\eta_s}}(v,v)}=\sqrt{-1}\frac{i\frac{\partial h_{\widehat{\eta_s}}}{\partial s}(v,v)}{h_{\widehat{\eta_s}}(v,v)}\\
&= \sqrt{-1}\frac{i\frac{\partial h_s}{\partial s}(v,v)+k^{-1}i\frac{\partial(h_s\cdot\widehat{\eta_s})}{\partial s}(v,v)}{h_s(v,v)+k^{-1}h_s(\widehat{\eta_s}(v),v)}\\
&= \sqrt{-1}\frac{i\frac{\partial h_s}{\partial s}(v,v)+k^{-1}i\frac{\partial h_s}{\partial s}(\widehat{\eta_s}(v),v)+k^{-1}ih_s(v,\frac{\partial\widehat{\eta_s}}{\partial s}v)}{h_s(v,v)}\left(1+k^{-1}\frac{h_s(\widehat{\eta_s}(v),v)}{h_s(v,v)}\right)^{-1}\\
&= \sqrt{-1}\frac{h_s(ih_s^{-1}\frac{\partial h_s}{\partial s}v,v)}{h_s(v,v)}\left(1+k^{-1}\Phi_{h_s}(-i\widehat{\eta_s})\right)^{-1}([v])\\
&\quad +\sqrt{-1}\frac{k^{-1}h_s(ih_s^{-1}\frac{\partial h_s}{\partial s}\circ\widehat{\eta_s}(v),v)+k^{-1}h_s(i\frac{\partial\widehat{\eta_s}}{\partial s}(v),v)}{h_s(v,v)}\left(1+k^{-1}\Phi_{h_s}(-i\widehat{\eta_s})\right)^{-1}([v])\\
&= \Phi_{h_s}\left(ih_s^{-1}\frac{\partial h_s}{\partial s}\right)([v])\cdot\sum_{i=0}^{\infty}k^{-i}(\Phi_{h_s}(i\widehat{\eta_s}))^i([v]))\\
&\quad +k^{-1}\left(\Phi_{h_s}\left(ih_s^{-1}\frac{\partial h_s}{\partial s}\circ\widehat{\eta_s}\right)([v])+\Phi_{h_s}\left(i\frac{\partial\widehat{\eta_s}}{\partial s}\right)([v])\right)\sum_{i=0}^{\infty}k^{-i}(\Phi_{h_s}(i\widehat{\eta_s}))^i([v]).
\end{aligned}
$$

Now since h_s follows the HYM flow, namely

$$
ih_s^{-1}\partial_s h_s=2(\Lambda_\omega F_{h_s}-i\mu(\mathcal{E})Id_E),
$$

in particular

$$
\Phi_{h_{\widehat{\eta_s}}}\left(ih_{\widehat{\eta_s}}^{-1}\frac{\partial h_{\widehat{\eta_s}}}{\partial s}\right)
$$

$$\begin{aligned}(5.49)\qquad &= 2\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s}) + 2\mu(\mathcal{E}) + k^{-1}\Phi_{h_s}\left(i\frac{\partial\widehat{\eta}_s}{\partial s}\right) \\ &+k^{-1}\left(2\Phi_{h_s}\left(\Lambda_{\omega_\Sigma}F_{h_s}\circ\widehat{\eta}_s\right) + 2\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s})\cdot\Phi_{h_s}\left(i\widehat{\eta}_s\right)\right) + \mathcal{O}(k^{-2}).\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}\Phi_{h_{\widehat{\eta}_s}}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}})([v]) &= \sqrt{-1}\frac{h_{\widehat{\eta}_s}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}v,v)}{h_{\widehat{\eta}_s}(v,v)} = \sqrt{-1}\frac{h_s(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}v,v) + k^{-1}h_s(\widehat{\eta}_s\circ\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}v,v)}{h_s(v,v)+k^{-1}h_s(\widehat{\eta}_s(v),v)} \\ &= \left(\Phi_{h_s}\left(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}\right)([v]) + k^{-1}\Phi_{h_s}\left(\widehat{\eta}_s\circ\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}\right)([v])\right)\sum_{i=0}^{\infty}k^{-i}(\Phi_{h_s}(i\widehat{\eta}_s))^i([v]),\end{aligned}$$

so that

$$\begin{aligned}&\Phi_{h_{\widehat{\eta}_s}}(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}) \\ (5.50) = \quad &\Phi_{h_s}\left(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}\right) + k^{-1}\left(\Phi_{h_s}\left(\widehat{\eta}_s\circ\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}\right) + \Phi_{h_s}\left(\Lambda_{\omega_\Sigma}F^\circ_{h_{\widehat{\eta}_s}}\right)\Phi_{h_s}\left(i\widehat{\eta}_s\right)\right) + \mathcal{O}(k^{-2}).\end{aligned}$$

By the construction of the Chern connection $A_{h_{\widehat{\eta}_s}} = (\overline{\partial}_{\mathcal{E}}, h_{\widehat{\eta}_s})$, we have

$$A_{h_{\widehat{\eta}_s}} = \overline{\partial}_{\mathcal{E}} + h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s},$$

where we regard $h_{\widehat{\eta}_s}$ as a complex anti-linear map

$$h_{\widehat{\eta}_s} : E\to E^*,$$

and for a section σ of E

$$\left(h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s}\right)(\sigma) = h^{-1}_{\widehat{\eta}_s}\left(\overline{\partial}_{\mathcal{E}^*}\left(h_{\widehat{\eta}_s}(\sigma)\right)\right)$$

We then have

$$\begin{aligned}A_{h_{\widehat{\eta}_s}} - A_{h_s} &= h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s} - h_s^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s \\ &= h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s} - \partial_{(\mathcal{E},h_s)}\end{aligned}$$

Therefore

$$\begin{aligned}F_{A_{h_{\widehat{\eta}_s}}} &= F_{h_{\widehat{\eta}_s}} = F_{h_s} + d_{A^{End(E)}_{h_s}}\left(h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s} - \partial_{(\mathcal{E},h_s)}\right) \\ &+\left(h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s} - \partial_{(\mathcal{E},h_s)}\right)^2 \\ &= F_{h_s} + \overline{\partial}_{End(E)}\left(h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_{\widehat{\eta}_s} - \partial_{(\mathcal{E},h_s)}\right).\end{aligned}$$

since there are no forms of degree $(2,0)$ or $(0,2)$.

Now we have

$$\begin{aligned}&h^{-1}_{\widehat{\eta}_s}\circ\overline{\partial}_{E^*}\circ h_{\widehat{\eta}_s} \\ = \quad &(h_s + k^{-1}h_s\circ\widehat{\eta}_s)^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s + k^{-1}(h_s+k^{-1}h_s\circ\widehat{\eta}_s)^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s\circ\widehat{\eta}_s \\ = \quad &\left(\sum_{i=0}^{\infty}(-1)^ik^{-i}(\widehat{\eta}_s)^i\right)\circ h_s^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s + k^{-1}\left(\sum_{i=0}^{\infty}(-1)^ik^{-i}(\widehat{\eta}_s)^i\right)\circ h_s^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s\circ\widehat{\eta}_s \\ = \quad &h_s^{-1}\circ\overline{\partial}_{\mathcal{E}^*}\circ h_s + k^{-1}\left(\partial_{(\mathcal{E},h)}\circ\widehat{\eta}_s - \widehat{\eta}_s\circ\partial_{(\mathcal{E},h)}\right) \\ &+\left(\Xi\left(\widehat{\eta}_s\right) + k^{-1}\widehat{\eta}_s - 1\right)\circ\partial_{(\mathcal{E},h)} + k^{-1}\circ\left(\sum_{i=1}^{\infty}(-1)^ik^{-i}(\widehat{\eta}_s)^i\right)\circ\partial_{(\mathcal{E},h)}\circ\widehat{\eta}_s\end{aligned}$$

$$
\begin{aligned}
&= \partial_{(\mathcal{E},h_s)} + k^{-1}\left(\partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s\right) + k^{-1}\left(\sum_{i=1}^{\infty}(-1)^i k^{-i}(\widehat{\eta}_s)^i\right)\circ \partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s \\
&= \partial_{(\mathcal{E},h_s)} + k^{-1}\left(\partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s\right) + \mathcal{O}(k^{-2}).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
F_{A_{h_{\widehat{\eta}_s}}} &= F_{h_s} + k^{-1}\left(\overline{\partial}_{End(E)}\partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s\right) + \overline{\partial}_{End(E)}\left(k^{-1}\left((\sum_{i=1}^{\infty}(-1)^i k^{-i}(\widehat{\eta}_s)^i\right)\circ \partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s\right) \\
&= F_{h_s} + k^{-1}\left(\overline{\partial}_{End(E)}\partial_{(End(\mathcal{E}),h_s)}\widehat{\eta}_s\right) + \mathcal{O}(k^{-2}), \qquad (5.51)
\end{aligned}
$$

and

$$
\Lambda_{\omega_\Sigma} F_{A_{h_{\widehat{\eta}_s}}} = \Lambda_{\omega_\Sigma} F_{h_s} - k^{-1}\left(i\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right) + \mathcal{O}(k^{-2}). \qquad (5.52)
$$

Finally we get

$$
\begin{aligned}
\Phi_{h_{\widehat{\eta}_s}}\left(\Lambda_{\omega_\Sigma} F_{h_{\widehat{\eta}_s}}\right) &= \Phi_{h_s}\left(\Lambda_{\omega_\Sigma} F_{h_s}\right) - k^{-1}\left(\Phi_{h_s}\left(i\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right)\right) \\
&\quad + k^{-1}\left(\Phi_{h_s}\left(\Lambda_{\omega_\Sigma} F_{h_s}\right)\Phi_{h_s}\left(i\widehat{\eta}_s\right) + \Phi_{h_s}\left(\widehat{\eta}_s \circ \Lambda_{\omega_\Sigma} F_{h_s}\right)\right) + \mathcal{O}(k^{-2}). \qquad (5.53)
\end{aligned}
$$

At a formal level we therefore get:

$$
\begin{aligned}
&\frac{\partial \widehat{\omega}''_{k,1}(s(t))}{\partial t} + i\overline{\partial}_J\partial_J\left(Scal\left(\widehat{\omega}''_{k,1}(s(t))\right)\right) \\
= \; &\frac{\partial \widehat{\omega}'_{k,1}(s(t))}{\partial t} + rk^{-1}\left(\Phi_{h_{\widehat{\eta}_s}}\left(ih_{\widehat{\eta}_s}^{-1}\frac{\partial h_{\widehat{\eta}_s}}{\partial s}\right) - \Phi_{h_s}\left(ih_s^{-1}\frac{\partial h_s}{\partial s}\right)\right) \\
&+ i\overline{\partial}_J\partial_J\left(-k^{-1}\left(2r(\Phi_{h_{\widehat{\eta}_s}}(\Lambda_{\omega_\Sigma} F_{h_{\widehat{\eta}_s}})\right)\right) \\
&+ i\overline{\partial}_J\partial_J\left(k^{-2}(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)) + 2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_{h_{\widehat{\eta}_s}}(\Lambda_{\omega_\Sigma} F_{h_{\widehat{\eta}_s}}))\right) \\
&+ i\overline{\partial}_J\partial_J\left(k^{-2}\left(\Psi_{\Sigma,2}(t) + \Psi_{\Phi_{h_{\widehat{\eta}_s}},2}(t) + \Psi_{\perp,2}(t)\right)\right) \\
&+ i\overline{\partial}_J\partial_J\left(\sum_{l=3} k^{-l}(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_{h_{\widehat{\eta}_s}},l}(s) + \Psi_{\perp,l}(s))\right) \\
= \; &\frac{\partial \widehat{\omega}'_{k,1}(s(t))}{\partial t} + k^{-2}2ri\overline{\partial}_J\partial_J\left(\Phi_{h_s}(\Lambda_{\omega_\Sigma} F_{h_s})\cdot\Phi_{h_s}\left(i\widehat{\eta}_s\right) + \Phi_{h_s}\left(\Lambda_{\omega_\Sigma} F_{h_s}\circ\widehat{\eta}_s\right) + \Phi_{h_s}\left(\frac{1}{2}i\frac{\partial\widehat{\eta}_s}{\partial s}\right)\right) \\
&- i\overline{\partial}_J\partial_J\left(k^{-1}\left(2r(\Phi_{h_s}\left(\Lambda_{\omega_\Sigma} F_{h_s}\right))\right)\right) + i\overline{\partial}_J\partial_J\left(k^{-2}(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)) + 2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_{h_s}(\Lambda_{\omega_\Sigma} F_{h_s}))\right) \\
&+ i\overline{\partial}_J\partial_J\left(k^{-2}\left(\Psi_{\Sigma,2}(t) + \Psi_{\Phi_{h_s},2}(t) + \Psi_{\perp,2}(t)\right)\right) \\
&+ i\overline{\partial}_J\partial_J\left(\sum_{l=3} k^{-l}(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_{h_s},l}(s) + \Psi_{\perp,l}(s))\right) \\
&+ k^{-2}2ri\overline{\partial}_J\partial_J\left(\Phi_{h_s}\left(i\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right)\right) \\
&- k^{-2}2ri\overline{\partial}_J\partial_J\left(\Phi_{h_s}\left(\Lambda_{\omega_\Sigma} F_{h_s}\right)\Phi_{h_s}\left(i\widehat{\eta}_s\right) + \Phi_{h_s}\left(\widehat{\eta}_s\circ\Lambda_{\omega_\Sigma} F_{h_s}\right)\right) + \mathcal{O}(k^{-3}) \\
= \; &\frac{\partial \widehat{\omega}'_{k,1}(s(t))}{\partial t} + i\overline{\partial}_J\partial_J Scal\left(\widehat{\omega}'_{k,1}(s(t))\right) + k^{-2}2r\left(\Phi_{h_s}\left(i\left(\frac{1}{2}\frac{\partial\widehat{\eta}_s}{\partial s} + \Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right) + [\Lambda_{\omega_\Sigma} F_{h_s}, \widehat{\eta}_s]\right)\right) \\
&+ \mathcal{O}(k^{-3})
\end{aligned}
$$

$$\begin{aligned}
&= k^{-2}i\overline{\partial}_J\partial_J\left((2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_{h_s}(\Lambda_{\omega_\Sigma}F_{h_s})+\Psi_{\Phi_{h_s},2}(t)+\Psi_{\perp,2}(t))\right)\\
&\quad +k^{-2}2ri\overline{\partial}_J\partial_J\left(\Phi_{h_s}\left(i\left(\frac{1}{2}\frac{\partial\widehat{\eta}_s}{\partial s}+\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right)+[\Lambda_{\omega_\Sigma}F_{h_s},\widehat{\eta}_s]\right)\right)+\mathcal{O}(k^{-3}).
\end{aligned}$$

Now recall that

$$\widehat{\omega}_{k,1}^{''}(s(t))=\widetilde{g}_s^*(\omega_{k,1}^{''}(s(t))).$$

By Lemma 4.12 we obtain

$$\begin{aligned}
&2rk^{-1}\left(\frac{\partial\omega_{k,1}^{''}(s(t))}{\partial s}+\mathcal{L}_{V_s}\left(\omega_{k,1}^{''}(s(t))\right)\right)+i\overline{\partial}_{J_s}\partial_{J_s}\left(Scal\left(\omega_{k,1}^{''}(s(t))\right)\right)\\
&= k^{-2}i\overline{\partial}_{Js}\partial_{J_s}\left((2r\Delta_{\omega_\Sigma}\Theta(w)\Phi_h(\Lambda_{(\omega_\Sigma}F_{A_s})+\Psi_{\Phi_h,2}(s(t))+\Psi_{\perp,2}(s(t)))\right)\\
&\quad +k^{-2}2ri\overline{\partial}_{J_s}\partial_{J_s}\left(\Phi_h\left(i(g_s^{-1})^*\left(\frac{1}{2}\frac{\partial\widehat{\eta}_s}{\partial s}+\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right)+(g_s^{-1})^*\left([\Lambda_{\omega_\Sigma}F_{h_s},\widehat{\eta}_s]\right)\right)\right)+\mathcal{O}(k^{-3}).
\end{aligned}$$

Using the formulae

$$\widehat{\eta}_s = g_s^{-1}\circ\eta_s\circ g_s, \Lambda_{\omega_\Sigma}F_{h_s}=g_s^{-1}\circ\Lambda_{\omega_\Sigma}F_{A_s}\circ g_s$$

and Equation 3.11 one easily calculates that

$$\begin{aligned}
(g_s^{-1})^*\left(\frac{1}{2}i\frac{\partial\widehat{\eta}_s}{\partial s}\right) &= \frac{1}{2}i\frac{\partial\eta_s}{\partial s}+\frac{1}{2}\left[\eta_s,\Lambda_{\omega_\Sigma}F_{A_s}\right]\\
(g_s^{-1})^*\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\eta_s) &= \Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s),\\
\text{and } (g_s^{-1})^*\left([\Lambda_{\omega_\Sigma}F_{h_s},\widehat{\eta}_s]\right) &= [\Lambda_{\omega_\Sigma}F_{A_s},\eta_s],
\end{aligned}$$

so that

$$\begin{aligned}
&i(g_s^{-1})^*\left(\frac{1}{2}\frac{\partial\widehat{\eta}_s}{\partial s}+\Delta_{\partial_{h_s}^{End(\mathcal{E})}}(\widehat{\eta}_s)\right)+(g_s^{-1})^*\left([\Lambda_{\omega_\Sigma}F_{h_s},\widehat{\eta}_s]\right)\\
&= i\left(\frac{1}{2}\frac{\partial\eta_s}{\partial s}+\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)\right)+\frac{1}{2}[\Lambda_{\omega_\Sigma}F_{A_s},\eta_s],
\end{aligned}$$

and we obtain

$$\begin{aligned}
&rk^{-1}\left(\frac{\partial\omega_{k,1}^{''}(s(t))}{\partial s}+\mathcal{L}_{V_s}\left(\omega_{k,1}^{''}(s(t))\right)\right)+i\overline{\partial}_{J_s}\partial_{J_s}\left(Scal\left(\omega_{k,1}^{''}(s(t))\right)\right)\\
&= k^{-2}\left(i\overline{\partial}_{Js}\partial_{J_s}\left(r\Phi_h\left(i\frac{\partial\eta_s}{\partial s}+2i\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)+2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s}+[\Lambda_{\omega_\Sigma}F_{A_s},\eta_s]+\alpha(s)\right)\right)\right)\\
&\quad +k^{-2}\left(i\overline{\partial}_{Js}\partial_{J_s}\Psi_{\perp,2}(s(t)))\right)+\mathcal{O}(k^{-3}),
\end{aligned}$$

where $\alpha(s)$ is a family of endomorphisms such that

$$\Phi_h\left(r\alpha(s)\right)=\Psi_{\Phi_h,2}(s(t)). \tag{5.54}$$

Recall the Bochner-Kodaira-Nakano identity, which applied to the induced connections $A_s^{End(E)}$ on $End(E)$, gives an equality

$$\begin{aligned}
\Delta_{A_s^{EndE}}(\eta_s) &= 2\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)+i\left[F_{A_s},\Lambda_\omega^{EndE}\right](\eta_s)\\
&= 2\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)-\left(i\Lambda_\omega^{EndE}F_{A_s}\right)(\eta_s)\\
&= 2\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)-i\left[\Lambda_\omega F_{A_s},\eta_s\right],
\end{aligned}$$

or

$$i\Delta_{A_s^{EndE}}(\eta_s)=2i\Delta_{\partial_h^{End(\mathcal{E}_s)}}(\eta_s)+[\Lambda_\omega F_{A_s},\eta_s]$$

and so this last quantity is equal to

$$k^{-2}\left(i\overline{\partial}_{J_s}\partial_{J_s}r\Phi_h\left(i\left(\frac{\partial\eta_s}{\partial s}+\Delta_{A_s^{EndE}}(\eta_s)\right)+2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s}+\alpha(s)\right)+\Psi_{\perp,2}(s(t)))\right)+\mathcal{O}(k^{-3}).$$

We now define the one parameter family η_s. We note that since by Theorem 3.5 the limiting holomorphic bundle

$$\mathcal{E}_\infty\cong Gr(\mathcal{E})=\oplus_i\mathcal{Q}_j$$

splits as a direct sum of of stable bundles Q_j, by Lemma 3.4 an element of $\ker\Delta_{A_\infty^{EndE}}$ is of the form

$$\sum_j c_{j,\infty}Id_{\mathcal{Q}_j},$$

and so if

$$\Phi_h\left(r\alpha_\infty\right)=\Psi_{\Phi_h,2,\infty},$$

then we may write

$$-\left(2\Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty}+\alpha_\infty\right)=\beta_\infty+\sum_j c_{j,\infty}Id_{\mathcal{Q}_j},$$

where

$$\beta_\infty\perp\ker\Delta_{A_\infty^{EndE}},$$

where here $\perp$ means $L^2(g_\Sigma)$ orthogonal. Since the bundle $\mathcal{E}$, which is isomorphic to $\mathcal{E}_s$ for all s, is simple, we

$$\ker\Delta_{A_s^{EndE}}\subset\ker\Delta_{A_\infty^{EndE}}$$

for all s, because if $c_{j,\infty}=c_\infty$ for all j, then

$$\sum_j c_\infty Id_{\mathcal{Q}_j}=c_\infty Id_E$$

where we use the fact that, as smooth vector bundles

$$Gr(\mathcal{E})\simeq E.$$

Therefore we also have

$$\beta_\infty\perp\ker\Delta_{A_s^{EndE}}$$

for all s. Then for each s there is a solution $G_s(\beta_\infty)$ to the elliptic equation

$$\Delta_{A_s^{EndE}}\left(G_s(i\beta_\infty)\right)=i\beta_\infty, \tag{5.55}$$

where G_s is the Green's operator for $\Delta_{A_s^{EndE}}$.

We may similarly write

$$-(2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s}+\alpha(s))=\beta(s)+\sum_j c_j(s)Id_{Q_j}$$

where

$$\sum_i c_i(s)Id_{Q_i}=pr_{\ker\Delta_{A_\infty^{EndE}}}\left(-(2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s}+\alpha(s)\right),$$

$$\beta(s)\perp\ker\Delta_{A_\infty^{EndE}},$$

and $c_i(s)$ is a constant for each i and s.

Note that we may solve the system of ordinary differential equations

$$\begin{aligned}\frac{d\overline{\eta}_s}{dt}&=-i\sum_j c_j(s)Id_{Q_j}\\ \overline{\eta}_0&=0,\end{aligned} \tag{5.56}$$

for all time. Now fix a large positive number S and consider the cutoff function $g_S(s)$ such that $g_S \equiv 1$ on the interval $[0,S]$ and $g_S \equiv 0$ on $[2S,\infty)$. By construction then

$$g_S(s)\cdot \overline{\eta}_s \in W_{p,q,w_\varepsilon(s)}(g_\Sigma,h),$$

and still solves equation on the interval $[0,S]$.

Then we define $\widetilde{\eta}_s$ to be the solution to the initial value equation

$$\frac{\partial \widetilde{\eta}_s}{\partial s} + \Delta_{A_s^{EndE}}(\widetilde{\eta}_s) = -i(\beta(s)-\beta_\infty) + \partial_s\left(G_s(i\beta_\infty)\right) - g_S(s)\cdot \Delta_{A_s}(\overline{\eta}_s) \tag{5.57}$$
$$\widetilde{\eta}_0 = G_0\left(i\beta_\infty\right)$$

which we obtain from Theorem 7.10.

We briefly explain why this theorem applies. By construction we have

$$(\beta(s)-\beta_\infty)\perp \ker \Delta_{A_s^{EndE}}, \ker \Delta_{A_\infty^{EndE}}$$

for all s. Similarly, since

$$G_s(i\beta_\infty)\perp \ker \Delta_{A_s^{EndE}}, \ker \Delta_{A_\infty^{EndE}}$$

for all s by construction, for the tangent vectors we also have

$$\partial_s\left(G_s(i\beta_\infty)\right)\perp \ker \Delta_{A_s^{EndE}}, \ker \Delta_{A_\infty^{EndE}}$$

for all s, since for example, if σ is any element of $\ker \Delta_{A_\infty^{EndE}}$

$$\left\langle \left(G_s(i\beta_\infty)\right),\sigma\right\rangle_{L^2(g_\Sigma)} = 0$$

and so

$$0 = \partial_s\left(\left\langle \left(G_s(i\beta_\infty)\right),\sigma\right\rangle_{L^2(g_\Sigma)}\right) = \left\langle \left(\partial_s G_s(i\beta_\infty)\right),\sigma\right\rangle_{L^2(g_\Sigma)}.$$

Since $g_S(s)$ is 0 for s sufficiently large, the right hand side of our equation is therefore orthogonal to the kernels for large s.

Moreover, since by the proof of Lemma 4.9 $\Psi_{\Phi_h,2}(s(t))\to\Psi_{\Phi_h,2,\infty}$, smoothly at a rate of $\frac{1}{\sqrt{t}}$ and all time derivatives of $\Psi_{\Phi_h,2}(s(t))$ converge to zero at the same rate, by Lemma 4.15 $\alpha(s)\to\alpha_\infty$ smoothly at a rate of $\frac{1}{\sqrt{t}}$, and all time derivatives $\alpha(s)$ converge to zero at the same rate. Similarly

$$\begin{aligned}
&\left\|\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s} - \Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty}\right\|_{C^m(g_\Sigma,h)}\\
= \;& \left\|\Delta_{\omega_\Sigma}\Theta(w)\cdot\left(\Lambda_{\omega_\Sigma}F_{A_s}-\Lambda_{\omega_\Sigma}F_{A_\infty}\right) + \Delta_{\omega_\Sigma}(\Theta(w)-\Theta_\infty)\cdot\Lambda_{\omega_\Sigma}F_{A_\infty}\right\|_{C^m(g_\Sigma,h)}\\
\le \;& C\left(\left\|\Lambda_{\omega_\Sigma}F_{A_s}-\Lambda_{\omega_\Sigma}F_{A_\infty}\right\|_{C^m(g_\Sigma,h)} + \left\|\Delta_{\omega_\Sigma}(\Theta(w)-\Theta_\infty)\right\|_{C^m(g_\Sigma,h))}\right)\\
\le \;& \frac{C}{\sqrt{s}}
\end{aligned}$$

for all m, for all sufficiently large t, and in the same way

$$\left\|\partial_s^j\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s}\right\|_{C^m(g_\Sigma,h)} \le \frac{C}{\sqrt{s}}$$

for all m and $j\ge 1$ for all sufficiently large s.

Then $\beta(s)+\sum_j c_j(s)Id_{Q_j}$ may be estimated in this way as well, and by the continuity of the orthogonal projection operator, we have $\beta(s)\to\beta_\infty$ smoothly at a rate of $\frac{1}{\sqrt{s}}$, and all time derivatives of $\beta(s)$ converge to zero at the same rate. Therefore we have

$$i\left(\beta(s)-\beta_\infty\right)\in W_{4,p,q,w_\varepsilon(s)}(g_\Sigma,h).$$

Note that projecting onto the kernel, we we also obtain in particular that in fact

$$-\left(2\Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty}+\alpha_\infty\right) = \beta_\infty + c_\infty Id_E,$$

for some constant c_∞, since $c(t)Id_E$ must converge to $c_\infty Id_E$.

Furthermore, by the Bochner-Kodaira -Nakano identity we have:

$$\begin{aligned}
& \Delta_{A_s^{EndE}} - \Delta_{A_\infty^{EndE}} \\
= \; & 2i\Lambda_{\omega_\Sigma}\overline{\partial}_{End(\mathcal{E}_s)}\partial_{(End(\mathcal{E}_s),h)} - 2i\Lambda_{\omega_\Sigma}\overline{\partial}_{End(\mathcal{E}_\infty)}\partial_{(End(\mathcal{E}_\infty),h)} \\
& +i([\Lambda_{\omega_\Sigma}F_{A_\infty}, -] - [\Lambda_{\omega_\Sigma}F_{A_s}, -]) \\
= \; & 2i\Lambda_{\omega_\Sigma}(\overline{\partial}_{End(\mathcal{E}_\infty)} + a_s^{0,1})(\partial_{(End(\mathcal{E}_\infty),h)} + a_s^{1,0}) - 2i\Lambda_{\omega_\Sigma}\overline{\partial}_{End(\mathcal{E}_\infty)}\partial_{(End(\mathcal{E}_\infty),h)} \\
& +i([\Lambda_{\omega_\Sigma}F_{A_\infty} - \Lambda_{\omega_\Sigma}F_{A_s}, -] \\
= \; & 2i\Lambda_{\omega_\Sigma}\left(\overline{\partial}_{End(\mathcal{E}_\infty)} \circ a_s^{1,0} + a_s^{0,1} \circ \partial_{(End(\mathcal{E}_\infty),h)} + a_s^{0,1} \wedge a_s^{1,0},\right) \\
& +i([\Lambda_{\omega_\Sigma}F_{A_\infty} - \Lambda_{\omega_\Sigma}F_{A_s}, -]
\end{aligned} \tag{5.58}$$

where $a_s^{0,1}$ and $a_s^{1,0}$ converge smoothly to zero at a rate of $1/\sqrt{s}$. We therefore obtain bounds of the form

$$\left\|\partial_s^j\left(\Delta_{A_s^{EndE}} - \Delta_{A_\infty^{EndE}}\right)\right\|_{C^m(g_\Sigma,h)} \leq \frac{C}{\sqrt{s}}$$

for all m and j and all sufficiently large t. Since we have

$$\Delta_{A_s^{EndE}} \circ G_s = G_s \circ \Delta_{A_s^{EndE}} = Id_{(\ker \Delta_{A_s^{EndE}})^\perp},$$

and since $\Delta_{A_s^{EndE}}$ converges to $\Delta_{A_\infty^{EndE}}$, we also have $G_s \to G_\infty$, smoothly, and all time derivatives go to zero at a rate of $1/\sqrt{t}$. Moreover

$$G_s - G_\infty = G_\infty \circ \left(\Delta_{A_\infty^{EndE}} - \Delta_{A_s^{EndE}}\right) \circ G_s,$$

so we obtain a bound

$$\left\|\partial_s^j\left(G_s - G_\infty\right)\right\|_{C^m(g_\Sigma,h)} \leq \frac{C}{\sqrt{s}}$$

for all m and j and all sufficiently large t. In particular we obtain

$$\partial_s\left(G_s(i\beta_\infty)\right) \in W_{4,p,q,w_\varepsilon(s)}(g_\Sigma, h).$$

Therefore since $g_S(s) \cdot \Delta_{A_s}(\overline{\eta}_s)$ is 0 for large s, the result applies.

Now we may define η_s by

$$\eta_s = \widetilde{\eta}_s + G_s(-i\beta_\infty) + g_S(s) \cdot \overline{\eta}_s, \tag{5.59}$$

which by definition then satisfies the equation

$$\frac{\partial \eta_s}{\partial s} + \Delta_{A_s^{EndE}}(\eta_s) = -i\beta(s) + \frac{\partial}{\partial s}\left(g_S(s) \cdot \overline{\eta}_s\right), \tag{5.60}$$

$$\eta_0 = 0.$$

Notice that on the interval $[0, S]$, the right hand side of this equation is exactly

$$i\left(2\Delta_{\omega_\Sigma}\Theta(w)\Lambda_{\omega_\Sigma}F_{A_s} + \alpha(s)\right).$$

Remark 5.9. We remark that we should really write η_s^S for η_s, since we have really constructed a one parameter family of paths, with each path depending on our choice of cut-off function. However, here and in the sequel we will drop this piece of notation. Note also that the limit of the η_s^S at infinity is independent of S, since the cut-off function vanish for sufficiently large s.

Moreover, by the parabolic Sobolev theory we obtain an estimate of the form

$$\begin{aligned}
& \|\widetilde{\eta}_s\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
= \;& \|\eta_s - G_s(-i\beta_\infty) - g_S(s)\cdot\overline{\eta}_s\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
\leq \;& C\,\|G_0(i\beta_\infty)\|_{L^2_{4p+2}(g_\Sigma,h)} \\
& +C\,\|i(\beta(s)-\beta_\infty) + \partial_s\left(G_s(i\beta_\infty)\right) - g_S(s)\cdot\Delta_{A_s}(\overline{\eta}_s)\,\|_{W_{4,p,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
= \;& \mathcal{O}(1).
\end{aligned}$$

so that if we write $\eta_\infty = G_\infty(-i\beta_\infty)$, we have

$$\begin{aligned}
& \|\eta_s - \eta_\infty\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
= \;& \|\eta_s - G_s(-i\beta_\infty) + G_s(-i\beta_\infty) - G_\infty(i\beta_\infty) + g_S(s)\cdot\overline{\eta}_s - g_S(s)\cdot\overline{\eta}_s\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
\leq \;& \|\widetilde{\eta}_s\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} + \|G_s(-i\beta_\infty) - G_\infty(-i\beta_\infty)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
& + \|g_S(s)\cdot\overline{\eta}_s\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_\Sigma,h)} \\
= \;& \mathcal{O}(1).
\end{aligned} \tag{5.61}$$

Formally, on the interval $[0,S]$ we obtain:

$$\begin{aligned}
& rk^{-1}\left(\frac{\partial\omega''_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}\left(\omega''_{k,1}(s(t))\right)\right) + i\overline{\partial}_{J_s}\partial_{J_s}\left(Scal\left(\omega''_{k,1}(s(t))\right)\right) \\
= \;& k^{-2}\left(i\overline{\partial}_{Js}\partial_{J_s}\left(r\Phi_h\left(i\left(i\left(2\Delta_{\omega_\Sigma}\Theta(w)\Lambda_{\omega_\Sigma}F_{A_s}+\alpha(s)\right)\right)+2\Delta_{\omega_\Sigma}\Theta(w)\Lambda_{\omega_\Sigma}F_{A_s}+\alpha(s)\right)\right)\right)\right) \\
& +k^{-2} i\overline{\partial}_{J_s}\partial_{J_s}\Psi_{\perp,2}(s(t)) + \mathcal{O}(k^{-3}) \\
= \;& k^{-2}\left(i\overline{\partial}_J\partial_J\Psi_{\perp,2}(s(t))\right)\right) + \mathcal{O}(k^{-3}).
\end{aligned} \tag{5.62}$$

Pulling back by $\widetilde{g}_s$ again, so we get an analogue of equation 5.29

$$\frac{\partial\widehat{\omega}''_{k,1}(s(t))}{\partial t} + i\overline{\partial}_J\partial_J Scal\left(\widehat{\omega}''_{k,1}(s(t))\right) = k^{-2}\left(i\overline{\partial}_J\partial_J\Psi_{\perp,2}(s(t))\right)\right) + \mathcal{O}(k^{-3}), \tag{5.63}$$

for $s\in[0,S]$. Then we have formally eliminated $\Psi_{\Phi_{h_s},2}(s(t))$ (at least on this interval, but again, note that the interval is arbitrary).

Now we define

$$\begin{aligned}
H(\omega''_{k,1}(s(t)) \;=\;& H(\omega'_{k,1}(s(t)) - k^{-2}\Psi_{\Phi_h,2}(s(t)) + 2rk^{-2}\left(\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_s}\right)\Phi_h\left(i\eta_s\right)\right) \\
& -rk^{-2}\left(\Phi_h\left(i\Delta_{A_s^{EndE}}\left(\eta_s\right) + 2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s} - 2\Lambda_{\omega_\Sigma}F_{A_s}\circ\eta_s\right)\right) \\
& +2rk^{-1}\Phi_h(\Lambda_\omega F_{A_s})\left(\sum_{i=2}^{\infty}(-1)^i k^{-i}(\Phi_h\left(i\eta_s\right)^i\right)\right) \\
& +k^{-2}r\Phi_h\left(i\frac{\partial\eta_s}{\partial s} + 2\Lambda_{\omega_\Sigma}F_{A_s}\circ\eta_s)\right)\left(\sum_{i=1}^{\infty}(-1)^i k^{-i}(\Phi_h\left(i\eta_s\right)^i\right)\right).
\end{aligned} \tag{5.64}$$

Pulling back the formula 5.48 for $\frac{\partial\widehat{\omega}''_{k,1}(s(t))}{\partial s}$ by $(\widetilde{g}_s)^{-1}$ and also using equation 5.49 as well as equation 5.40, one may check that we obtain an analogue of equation 5.28, namely for $s\in[0,S]$:

$$\begin{aligned}
& rk^{-1}\left(\frac{\partial\omega''_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}\left(\omega''_{k,1}(s(t))\right)\right) \\
= \;& i\overline{\partial}_{J_s}\partial_{J_s}H(\omega''_{k,1}(s(t)).
\end{aligned} \tag{5.65}$$

By taking t to infinity, and noting that by the parabolic theory we have C^∞ convergence $\eta_s \to \eta_\infty$, with η_∞ as defined above, we obtain a fixed Kähler metric

$$\begin{aligned} \omega^{''}_{k,1,\infty} &= \omega^{'}_{k,1,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(k^{-1}\Xi\left(\eta_s\right)\right) \\ &= \omega\left(h_{\eta_\infty}, J_\infty\right) + k\omega_\Sigma + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\Theta_\infty \end{aligned} \tag{5.66}$$

on $\mathbb{P}(\mathcal{E}_\infty)$.

We analogously define

$$\begin{aligned} H(\omega^{''}_{k,1,\infty}) &= H(\omega^{'}_{k,1,\infty}) - k^{-2}\Psi_{\Phi_h,2,\infty} + 2rk^{-2}\left(\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\Phi_h\left(i\eta_\infty\right)\right) \\ &\quad -rk^{-2}\left(\Phi_h\left(i\Delta_{A^{EndE}_\infty}\left(\eta_\infty\right) + 2\Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty} - 2\Lambda_{\omega_\Sigma}F_{A_\infty}\circ\eta_\infty\right)\right) \\ &\quad +2rk^{-1}\Phi_h(\Lambda_\omega F_{A_\infty})\left(\sum_{i=2}^{\infty}(-1)^i k^{-i}(\Phi_h\left(i\eta_\infty\right)^i)\right) \\ &\quad +k^{-2}r\Phi_h\left(2\Lambda_{\omega_\Sigma}F_{A_\infty}\circ\eta_\infty\right)\left(\sum_{i=1}^{\infty}(-1)^i k^{-i}(\Phi_h\left(i\eta_\infty\right)^i)\right). \end{aligned} \tag{5.67}$$

Note that by definition, we have

$$i\Delta_{A^{End(E)}_\infty}\left(\eta_\infty\right) = \beta_\infty,$$

and therefore we may also write

$$H(\omega^{''}_{k,1,\infty}) = 2rk^{-1}\Phi_h\left(\Lambda_{\omega_\Sigma}F^\circ_{A_\infty} - \frac{k^{-1}}{2}\sum_j c_{j,\infty}Id_{Q_j}\right) + \mathcal{O}(k^{-2}). \tag{5.68}$$

The analogue of the expansion 5.34 for $Scal(\omega^{''}_{k,1,\infty})$ using (the analogue of) formula 5.50 is

$$\begin{aligned} Scal(\omega^{''}_{k,1,\infty}) &= Scal(\omega_{FS}(\mathbb{P}^{r-1})) + k^{-1}\left(Scal(\omega_\Sigma) - 2r\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\right) \\ &\quad +k^{-2}\left(r\Phi_h\left(i\Delta_{A^{End(E)}_\infty}\left(\eta_\infty\right) + 2\Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty}\right) + \Psi_{\Phi_h 2,\infty} + \Psi_{\perp,2,\infty}\right) \\ &\quad -2rk^{-2}\left(\Phi_h\left(\eta_\infty\circ\Lambda_{\omega_\Sigma}F_{A_\infty}\right) + \Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\cdot\Phi_h\left(i\eta_\infty\right)\right) + \mathcal{O}(k^{-3}) \\ &= Scal(\omega_{FS}(\mathbb{P}^{r-1})) + k^{-1}\left(Scal(\omega_\Sigma) - 2r\Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\right) \\ &\quad +rk^{-2}\left(\Phi_h\left(-\Lambda_{\omega_\Sigma}F_{A_\infty}\circ\eta_\infty - \eta_\infty\circ\Lambda_{\omega_\Sigma}F_{A_\infty} + c_\infty\right) + \Psi_{\perp,2,\infty}\right) \\ &\quad \Phi_h\left(\Lambda_{\omega_\Sigma}F_{A_\infty}\right)\cdot\Phi_h\left(i\eta_\infty\right) + \mathcal{O}(k^{-3}), \end{aligned}$$

so that

$$\begin{aligned} & Scal(\omega^{''}_{k,1,\infty}) + H\left(\omega^{''}_{k,1,\infty}\right) \\ =\; & Scal(\omega_{FS}(\mathbb{P}^{r-1})) + k^{-1}\left(Scal(\omega_\Sigma)\right) + k^{-2}\Psi_{\perp,2,\infty} \\ & +\mathcal{O}(k^{-3}). \end{aligned}$$

Notice that by equations 5.65, 5.62, 5.48 and 5.49, we have:

$$\begin{aligned} & Scal\left(\omega^{''}_{k,1}(s(t))\right) + H(\omega^{''}_{k,1}(s(t)) \\ =\; & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}Scal\left(\omega_\Sigma\right) \\ +\; & k^{-2}\left(\Psi_{\perp,2}(s(t)))\right) + \sum_{l=3}k^{-l}(\Psi_{\Sigma,l}(s) + \Psi_{\Phi_h,l}(s) + \Psi_{\perp,l}(s)), \end{aligned}$$

so that we obtain an analogue of equation 5.27, namely:

$$Scal\left(\omega^{''}_{k,1}(s(t))\right) + H(\omega^{''}_{k,1}(s(t)) - \left(Scal\left(\omega^{''}_{k,1,\infty}\right) + H(\omega^{''}_{k,1,\infty})\right)$$

$$(5.69)\qquad = k^{-2}\left(\Psi_{\perp,2}(s)-\Psi_{\perp,2,\infty}\right) + \sum_{l=3} k^{-l}\left(\left(\Psi_{\Sigma,l}(s)-\Psi_{\Sigma,l,\infty}\right)+\left(\Psi_{\Phi_h,l}(s)-\Psi_{\Phi_h,l,\infty}\right)+\left(\Psi_{\perp,l}(s)-\Psi_{\perp,l,\infty}\right)\right),$$

where we again note that for the moment we have kept the same notation for the terms of higher order, even though they have been modified.

Step 3: Correcting $\Psi_{\perp,2}(s)$ The final step is to correct $\Psi_{\perp,2}(s)$. We will do this by adding a function

$$\widehat{\Omega}(s(t)) := \tilde{g}_s^*(\Omega(s(t)))$$

for a 1-parameter family of functions $\Omega(s(t)) \in C^\infty(\mathbb{P}(E))$, where, as usual, this family will be determined later by solving a linear parabolic equation. We may finally define

$$\omega_{k,2}(s(t)) = \omega''_{k,1}(s(t)) + k^{-2} i\bar{\partial}_J\partial_J\left(\Omega(s(t))\right) \tag{5.70}$$

$$\widehat{\omega}_{k,2}(s(t)) = \widehat{\omega}''_{k,1}(s(t)) + k^{-2} i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right). \tag{5.71}$$

We will begin by calculating the scalar curvature of $\widehat{\omega}_{k,2}(t)$. We have

$$\begin{aligned} Scal\left(\widehat{\omega}_{k,2}(t)\right) &= \Lambda_{\widehat{\omega}_{k,2}(s(t))}\rho_{\widehat{\omega}_{k,2}(s(t))} \\ &= \Lambda_{\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)}\rho_{\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)} \\ &= \frac{\left(\rho_{\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)}\right)\wedge\left(\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^{r-1}}{\left(\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^{r}}. \end{aligned}$$

Now as in previous calculations we may write

$$\begin{aligned} &\rho_{\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)} \\ =\ &\rho_{\widehat{\omega}''_{k,1}(s(t))} + i\bar{\partial}_J\partial_J \log\left(\frac{\left(\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^r}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\right) \\ =\ &\rho_{\widehat{\omega}''_{k,1}(s(t))} + i\bar{\partial}_J\partial_J \log\left(1+k^{-2}\sum_{i=1}^{r}\binom{r}{i}k^{2-2i}\frac{\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-i}\wedge\left(i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^i}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\right) \\ =\ &\rho_{\widehat{\omega}''_{k,1}(s(t))} + k^{-2}i\bar{\partial}_j\partial_J\Delta_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t)) + \mathcal{O}(k^{-4}). \end{aligned}$$

Similarly, for any path of r-forms $\alpha(s)$ we have

$$\begin{aligned} &\frac{\alpha(s)}{\left(\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^r} \\ =\ &\frac{\alpha(s)}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\frac{1}{\left(1+k^{-2}\sum_{i=1}^{r}k^{2-2i}\binom{r}{i}\frac{\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-i}\wedge\left(i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)^i}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\right)} \\ =\ &\frac{\alpha(s)}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\left(1+k^{-2}\Delta_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t))+\mathcal{O}(k^{-4})\right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &Scal\left(\widehat{\omega}_{k,2}(s(t))\right) \\ =\ &Scal(\widehat{\omega}''_{k,1}(s(t))+k^{-2}i\bar{\partial}_j\partial_J\widehat{\Omega}(s(t)) \end{aligned}$$

$$\begin{aligned}
&= r\frac{\left(\rho_{\widehat{\omega}''_{k,1}(s(t))} + k^{-2}i\overline{\partial}_j\partial_J\Delta_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t)) + \mathcal{O}(k^{-4})\right)\wedge\left(\widehat{\omega}''_{k,1}(s(t)) + k^{-2}i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)\right)^{r-1}}{\left(\widehat{\omega}''_{k,1}(s(t)) + k^{-2}i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)\right)^{r}} \\
&= r\frac{\left(\rho_{\widehat{\omega}''_{k,1}(s(t))} + k^{-2}i\overline{\partial}_j\partial_J\Delta_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t))\right)\wedge\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-1}}{\left(\widehat{\omega}''_{k,1}(s(t)) + k^{-2}i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)\right)^{r}} \\
&+\frac{k^{-2}(r-1)\rho_{\widehat{\omega}''_{k,1}(s(t))}\wedge\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-2}\wedge i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)}{\left(\widehat{\omega}''_{k,1}(s(t)) + k^{-2}i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)\right)^{r}} + \mathcal{O}(k^{-4}) \qquad (5.72)\\
&= Scal\left(\widehat{\omega}''_{k,1}(s(t))\right) + k^{-2}\left(\Delta^2_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t)) - Scal\left(\widehat{\omega}''_{k,1}(s(t))\right)\left(\Delta_{\widehat{\omega}''_{k,1}(s(t))}\widehat{\Omega}(s(t))\right)\right) \\
&+k^{-2}r(r-1)\frac{\left(\rho_{\widehat{\omega}''_{k,1}(s(t))}\wedge\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-2}\wedge i\overline{\partial}_j\partial_J\widehat{\Omega}(s(t)\right)}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r}} \\
&= Scal\left(\widehat{\omega}''_{k,1}(s(t))\right) + k^{-2}\left(d_{\widehat{\omega}''_{k,1}(s(t))}Scal\left(\widehat{\Omega}(s(t))\right)\right) + \mathcal{O}(k^{-4}),
\end{aligned}$$

where in the last line we have used Lemma 2.5.

Notice that $d_{\widehat{\omega}''_{k,1}(s(t))}Scal\left(\widehat{\Omega}(s(t))\right)$ depends on k, so we will need to calculate this more explicitly to see what the k^{-2} term of this expansion is. We begin by expanding the Laplacian. We have

$$\begin{aligned}
\Delta_{\widehat{\omega}''_{k,1}(s(t))}\left(\widehat{\Omega}(s(t))\right) &= \Lambda_{\widehat{\omega}''_{k,1}(s(t))}\left(i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right) \\
&= r\frac{i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\wedge\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r-1}}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^{r}} \\
&= \frac{i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\wedge\left(\omega(h_s,J) + i\bar{\partial}_\Sigma\partial_\Sigma\Theta(w) + k^{-1}i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right))^{r-1}}{\left(\omega_k(h_s,J) + i\bar{\partial}_\Sigma\partial_\Sigma\Theta(w) + k^{-1}i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right))^{r}} \\
&= \frac{\left(i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)_{\mathcal{HH}}}{(\Phi_{h_s}(-\Lambda_{\omega_\Sigma}F_{h_s}) + \Delta_{\omega_\Sigma}\Theta(w) + k^{-1}\Delta_{\mathcal{H}}\Xi(\widehat{\eta}_s) + k)\omega_\Sigma} \qquad (5.73)\\
&+(r-1)\frac{\left(i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)\right)_{\mathcal{VV}}\wedge\left(\omega_{FS}(h_s) + k^{-1}\left(i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right)_{\mathcal{VV}}\right)^{r-2}}{\left(\omega_{FS}(h_s) + k^{-1}\left(i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right)_{\mathcal{VV}}\right)^{r-1}} \\
&= \Delta_{(\mathcal{V},h_s)}\left(\widehat{\Omega}(s(t))\right) + \mathcal{O}(k^{-1}).
\end{aligned}$$

We also have

$$\begin{aligned}
\rho_{\widehat{\omega}''_{k,1}(s(t))} &= \rho_{\widehat{\omega}_{k,1}(s(t))} \\
+i\bar{\partial}_J\partial_J\log&\left(\frac{(\Phi_{h_s}(-\Lambda_{\omega_\Sigma}F_{h_s}) + \Delta_{\omega_\Sigma}\Theta(w) + k^{-1}\Delta_{\mathcal{H}}\Xi(\widehat{\eta}_s) + k)\omega_\Sigma\wedge(r-1)\left(\omega_{FS}(h_s) + k^{-1}i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right)^{r-1}}{(r-1)\left(\omega_{FS}(h_s)\right)^{r-1}\wedge(\Phi_{h_s}(-\Lambda_{\omega_\Sigma}F_{h_s}) + k)\omega_\Sigma}\right) \\
&= \rho_{\widehat{\omega}_{k,1}(s(t))} + \mathcal{O}(k^{-1})
\end{aligned}$$

so that in the same way

$$\begin{aligned}
& r(r-1)\frac{\rho_{\widehat{\omega}''_{k,1}(s(t))}\wedge\widehat{\omega}''_{k,1}(s(t))^{r-2}\wedge i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}\\
= \;& r(r-1)\frac{(r\omega_{FS}(h_s)+(r\Phi_h(-\Lambda_\omega F_{A_t})+Scal(\omega_\Sigma))\,\omega_\Sigma)\wedge\widehat{\omega}''_{k,1}(s(t))^{r-2}\wedge i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)}{\left(\widehat{\omega}''_{k,1}(s(t))\right)^r}+\mathcal{O}(k^{-1})\\
= \;& (r-1)(r-2)\frac{r\omega_{FS}(h_s)\wedge i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)_{\mathcal{V}\mathcal{V}}\wedge\left(\omega_{FS}(h_s)+k^{-1}\left(i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right)_{\mathcal{V}\mathcal{V}}\right)^{r-3}}{\left(\omega_{FS}(h_s)+k^{-1}\left(i\bar{\partial}_J\partial_J\Xi(\widehat{\eta}_s)\right)_{\mathcal{V}\mathcal{V}}\right)^{r-1}}+\mathcal{O}(k^{-1})\\
= \;& r(r-1)(r-2)\frac{i\bar{\partial}_J\partial_J\left(\widehat{\Omega}(s(t))\right)_{\mathcal{V}\mathcal{V}}\wedge(\omega_{FS}(h_s))^{r-2}}{(\omega_{FS}(h_s))^{r-1}}+\mathcal{O}(k^{-1})\\
= \;& r(r-2)\Delta_{(\mathcal{V},h_s)}\left(\widehat{\Omega}(s(t))\right)+\mathcal{O}(k^{-1}).
\end{aligned}$$

Then finally we obtain

$$\begin{aligned}
& Scal\left(\widehat{\omega}_{k,2}(s(t))\right)\\
= \;& Scal\left(\widehat{\omega}''_{k,1}(s(t))\right)+k^{-2}\left(\Delta^2_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)-Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right)\Delta_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)\right)\\
& +k^{-2}\left(r(r-2)\Delta_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)\right)+\mathcal{O}(k^{-3})\\
& Scal\left(\widehat{\omega}''_{k,1}(s(t))\right)+k^{-2}\left(\Delta^2_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)-2r\Delta_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)\right)+\mathcal{O}(k^{-3}),
\end{aligned}$$

since

$$Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right)=2r(r-1).$$

We will write

$$\begin{aligned}
& \mathfrak{D}^*_{(\mathcal{V},h_s)}\mathfrak{D}_{(\mathcal{V},h_s)}\left(\widehat{\Omega}(s(t))\right)\\
: \;& =\Delta^2_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)-2r\Delta_{\mathcal{V}}\left(\widehat{\Omega}(s(t))\right)
\end{aligned}\tag{5.74}$$

so that

$$Scal\left(\widehat{\omega}_{k,2}(s(t))\right)=Scal\left(\widehat{\omega}''_{k,1}(s(t))\right)+k^{-2}\mathfrak{D}^*_{(\mathcal{V},h_s)}\mathfrak{D}_{(\mathcal{V},h_s)}\left(\widehat{\Omega}(s(t))\right)+\mathcal{O}(k^{-3}).\tag{5.75}$$

We observe that for any holmorphic structure on E giving rise to a vertical bundle $\mathcal{V}$, and any hermitian metric on E; $\mathfrak{D}^*_{(\mathcal{V},h)}\mathfrak{D}_{(\mathcal{V},h)}$ is an operator $C^\infty(\mathbb{P}(E))\to C^\infty(\mathbb{P}(E))$ which restricts to each fibre to be the operator $C^\infty(\mathbb{P}(E_x))\to C^\infty(\mathbb{P}(E_x))$ given by the Lichnerowicz operator $\mathfrak{D}^*_x\mathfrak{D}_x$ on the fibre associated to the Fubini-Study metric. That is,

$$\mathfrak{D}^*_{(\mathcal{V},h)}\mathfrak{D}_{(\mathcal{V},h)}(\phi)|_{\mathbb{P}(E_x)}=\mathfrak{D}^*_x\mathfrak{D}_x(\phi|_{\mathbb{P}(E_x)}).$$

It is also easy to see using the fact that $\widetilde{g}^*_s(\omega_{FS}(h_s,J))=\omega_{FS}(h,J_s)$ that

$$\mathfrak{D}^*_{(\mathcal{V},h_s)}\mathfrak{D}_{(\mathcal{V},h_s)}(\widetilde{g}^*_s(\Omega_s))=\widetilde{g}^*_s(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s)).$$

The fact that $\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}$ is also self-adjoint with respect to $\omega_{k,1}(s(t))$ follows from the self-adjointness of $\Delta_{(\mathcal{V}_s,h)}$ and $\Delta^2_{(\mathcal{V}_s,h)}$, which folllows from the self-adjointness of $\Delta_{\omega_{k,1}(s(t))}$ and $\Delta^2_{\omega_{k,1}(s(t))}$, and the equalities

$$\begin{aligned}
\Delta_{\omega_{k,1}(s(t))} &= \Delta_{(\mathcal{V}_s,h)}+\mathcal{O}(k^{-1})\\
\Delta^2_{\omega_{k,1}(s(t))} &= \Delta^2_{(\mathcal{V}_s,h)}+\mathcal{O}(k^{-1}),
\end{aligned}$$

(which follow from the exact same argument as for the metric $\widehat{\omega}''_{k,1}(s(t))$, by simply taking k to infinity.

As usual we may write

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{g}_s^*(\Omega_s) &= \frac{\partial}{\partial t}\widetilde{g}_s^*(\Omega_s) = \frac{\partial}{\partial s}\widetilde{g}_s^*(\Omega_s)\frac{\partial s}{\partial t} \\ &= 2rk^{-1}\widetilde{g}_s^*\left(\mathcal{L}_{V_s}(\Omega_s) + \frac{\partial}{\partial s}\Omega_s\right) \\ &= 2rk^{-1}\widetilde{g}_s^*(\mathcal{L}_{V_s}(\Omega_s)) + \widetilde{g}_s^*\left(\frac{\partial}{\partial t}\Omega_s\right)\end{aligned}$$

At a formal level, we therefore obtain for all $s \in [0, S]$:

$$\begin{aligned}&\frac{\partial\widehat{\omega}_{k,2}(s(t))}{\partial t} + i\bar{\partial}_J\partial_J\left(Scal\left(\widehat{\omega}_{k,2}(s(t))\right)\right) \\ =\;& \frac{\partial\widehat{\omega}''_{k,1}(s(t))}{\partial t} + i\bar{\partial}_J\partial_J\left(Scal\left(\widehat{\omega}_{k,2}(s(t))\right)\right) + k^{-2}i\bar{\partial}_J\partial_J\left(\frac{\partial}{\partial t}\widetilde{g}_s^*(\Omega(s))\right) \\ &+k^{-2}i\bar{\partial}_J\partial_J(\widetilde{g}_s^*\mathfrak{D}^*_{(\mathcal{V},h_s)}\mathfrak{D}_{(\mathcal{V},h_s)}((\Omega_s))) + \mathcal{O}(k^{-3}) \\ =\;& k^{-2}i\bar{\partial}_J\partial_J(\widetilde{g}_s^*\left(\Psi_{\perp,2}(s) + \frac{\partial}{\partial t}\Omega_s + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s))\right) \\ &+k^{-3}i\bar{\partial}_J\partial_J(\widetilde{g}_s^*(\mathcal{L}_{V_s}(\Omega_s)) + \mathcal{O}(k^{-3}).\end{aligned}$$

By the definition of Ω_s below, we will see that $\Omega_s \to \Omega_\infty$ smoothly as well, so that

$$2rk^{-3}i\bar{\partial}_J\partial_J(\widetilde{g}_s^*(\mathcal{L}_{V_s}(\Omega_s)) = \mathcal{O}(k^{-3})$$

and therefore also

$$\begin{aligned}&\frac{\partial\widehat{\omega}_{k,2}(s(t))}{\partial t} + i\bar{\partial}_J\partial_J\left(Scal\left(\widehat{\omega}_{k,2}(s(t))\right)\right) \\ =\;& k^{-2}i\bar{\partial}_J\partial_J(\widetilde{g}_s^*\left(\Psi_{\perp,2}(s) + \frac{\partial}{\partial t}\Omega_s + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s))\right) + \mathcal{O}(k^{-3})\end{aligned} \tag{5.76}$$

for all $s \in [0, S]$.

Given any endomorphism $F \in \mathfrak{su}(E)$, recall that the restriction of the vector field X_F to each fibre $\mathbb{P}(E_x)$ is holomorphic, and in fact

$$X_F|_{\mathbb{P}(E_x)} = \nabla\Phi_h(F)|_{\mathbb{P}(E_x)},$$

essentially by definition. In other words $\Phi_h(F)|_{\mathbb{P}(E_x)}$ is a holomorphy potential, and therefore we obtain in particular that

$$\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Phi_h(F))|_{\mathbb{P}(E_x)} = \mathfrak{D}^*_x\mathfrak{D}_x(\Phi_h(F)|_{\mathbb{P}(E_x)}) = 0$$

for all x, and therefore $\Phi_h(F) \in \ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)})$ for all s, or $\Phi_h(\mathfrak{su}(E)) \subset \ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)})$.

Another way to see this is that since the Ricci curvature of the Fubini-Study metric on $\mathbb{P}(E_x)$ is

$$Ric(\omega_{FS}) = ((r-1)+1)\,\omega_{FS} = r\omega_{FS}$$

(and in particular $Scal(\omega_{Fs})$ is constant), so that we have for any function ϕ

$$\begin{aligned}\mathfrak{D}^*_x\mathfrak{D}_x(\phi|_{\mathbb{P}(E_x)}) &= \Delta^2_{\overline{\partial}}(\phi|_{\mathbb{P}(E_x)}) + r\langle\omega_{FS}, i\partial\overline{\partial}(\phi|_{\mathbb{P}(E_x)})\rangle \\ &= \Delta^2_{\overline{\partial}}(\phi|_{\mathbb{P}(E_x)}) - r\Delta_{\overline{\partial}}(\phi|_{\mathbb{P}(E_x)})\end{aligned}$$

so a smooth eigenfunction of $\Delta_{\overline{\partial}}$ is in the kernel of $\mathfrak{D}^*_x\mathfrak{D}_x$ exactly when it is in the first eigenspace, that is, when it corresponds to the eigenvalue r. In otherwords, the smooth eigenfunctions of $\Delta_{\overline{\partial}}$

which are in the kernel of $\mathfrak{D}_x^*\mathfrak{D}_x$ are exactly the functions which are restrictions of functions in $\Phi_h(\mathfrak{su}(E))$. Since, by the spectral theorem, any L^2 function on $\mathbb{P}(E_x)$ has an orthonomal expansion in terms of the eigenfunctions of $\Delta_{\overline{\partial}}$, it follows that

$$\ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}) = \pi^*(C^\infty(\Sigma)) \oplus \Phi_h(\mathfrak{su}(E)), \tag{5.77}$$

for all s, and in fact this is true for any holomorphic structure, so in particular we also have

$$\ker(\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}) = \pi^*(C^\infty(\Sigma)) \oplus \Phi_h(\mathfrak{su}(E)). \tag{5.78}$$

By definition, the space $C_h^\infty(\mathbb{P}(E))_\perp$ consists of functions which are fibrewise L^2 orthogonal with respect to the Fubini-Study metric to the space on the right hand side of the above equality. Then since $\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}$ is self-adjoint with respect to $\omega_{k,1}(s(t))$ (and $\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}$ with respect to $\omega_{k,1,\infty}$, the decomposition

$$\begin{aligned} C^\infty(\mathbb{P}(E)) &= \ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}) \oplus C_h^\infty(\mathbb{P}(E))_\perp \\ &= \ker(\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}) \oplus C_h^\infty(\mathbb{P}(E))_\perp \\ &= \pi^*(C^\infty(\Sigma)) \oplus \Phi_h(\mathfrak{su}(E)) \oplus C_h^\infty(\mathbb{P}(E))_\perp \end{aligned}$$

is orthogonal with respect to the $L^2(g_{k,1}(s(t)))$ inner product for all s and with respect to the $L^2(g_{k,1,\infty})$ inner product.

In particular, for every s, $\Psi_{\perp,2,\infty}$ is $L^2(g_{k,1,\infty})$ orthogonal to $\ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)})$, so that we may solve the elliptic equation

$$\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}\left(G_s\left(\Psi_{\perp,2,\infty}\right)\right) = \Psi_{\perp,2,\infty}, \tag{5.79}$$

where G_s is the Green's operator for $\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}$. Note that $\Psi_{\perp,2}(s)$ is also $L^2(g_{k,1,\infty})$ orthogonal to $\ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)})$, for every s so that

$$-\left(\Psi_{\perp,2}(s) - \Psi_{\perp,2,\infty}\right) \perp_{L^2(g_{k,1,\infty})} \ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}).$$

for all s. Note also that for any $\phi \in \pi^*(C^\infty(\Sigma)) \oplus \Phi_h(\mathfrak{su}(E))$ we have

$$\begin{aligned} 0 &= \langle \phi, G_s\left(\Psi_{\perp,2,\infty}\right)\rangle_{L^2(g_{k,1,\infty})} \\ &\implies 0 = \frac{\partial}{\partial s}\langle \phi, G_s\left(\Psi_{\perp,2,\infty}\right)\rangle_{L^2(g_{k,1,\infty})} \\ &= \langle \phi, \partial_s G_s\left(\Psi_{\perp,2,\infty}\right)\rangle_{L^2(g_{k,1,\infty})}, \end{aligned}$$

so also

$$\partial_s G_s\left(\Psi_{\perp,2,\infty}\right) \perp_{L^2(g_{k,1,\infty})} \ker(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}).$$

for all s. By Lemma 5.5 we have

$$\Psi_{\perp,2}(s) - \Psi_{\perp,2,\infty} \in W_{4,p,,q,w_\varepsilon(s)}(g_{k,1,\infty}).$$

We will write G_∞ for the Green's operator for $\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}$. On the space $C_h^\infty(\mathbb{P}(E))_\perp$ we have

$$G_s - G_\infty = G_\infty \circ \left(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)} - \mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}\right) \circ G_s,$$

since

$$\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)} \circ G_s = G_\infty \circ \mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)} = Id_{C_h^\infty(\mathbb{P}(E))_\perp}.$$

One may easily show that

$$\left\|\partial_s^j\left(\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)} - \mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}\right)\right\|_{C^m(g_{k,1,\infty})} \leq \frac{C}{\sqrt{s}}$$

for all m and j, and all s sufficiently large, by using the expression for thes operators, and previous estimates. We therefore obtain $G_s \overset{C^\infty}{\to} G_\infty$, and in fact

$$\left\|\partial_s^j \left(G_s - G_\infty\right)\right\|_{C^m(g_{k,1,\infty})} \leq \frac{C}{\sqrt{s}}$$

for all m and j, and all s sufficiently large. In particular

$$\|\partial_t G_s\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = k^{-1}\|\partial_s G_s\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = \mathcal{O}(k^{-1/2}),$$

and in particular

$$\partial_t G_s\left(\Psi_{\perp,2,\infty}\right) \in W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty}).$$

By Theorem 7.10 we may therefore solve the parabolic equation

$$\frac{\partial}{\partial t}\widetilde{\Omega}_s + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\widetilde{\Omega}_s) = -\left(\Psi_{\perp,2}(s) - \Psi_{\perp,2,\infty}\right) + \partial_t G_s\left(\Psi_{\perp,2,\infty}\right), \tag{5.80}$$

$$\widetilde{\Omega}_0 = G_0\left(\Psi_{\perp,2,\infty}\right)$$

Now we define

$$\Omega_s = \widetilde{\Omega}_s + G_s\left(-\Psi_{\perp,2,\infty}\right)$$

so that Ω_s automatically satisfies the initial value equation

$$\frac{\partial}{\partial t}\Omega_s + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s) = -\Psi_{\perp,2}(s). \tag{5.81}$$

$$\Omega_0 = 0$$

We also obtain an estimate of the form

$$\begin{aligned}
\left\|\widetilde{\Omega}_s\right\|_{W_{4,p+1,,q,w_\varepsilon(s)}(g_\Sigma,h)} &= \|\Omega_s - G_s\left(-\Psi_{\perp,2,\infty}\right)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} \\
&\leq C\left\|G_0\left(\Psi_{\perp,2,\infty}\right)\right\|_{L^2_{4p+2}(g_{k,1,\infty})} \\
&\quad + C\left\|-\left(\Psi_{\perp,2}(s) - \Psi_{\perp,2,\infty}\right) + \partial_t G_s\left(\Psi_{\perp,2,\infty}\right)\right\|_{W_{4,p,,q,w_\varepsilon(s)}(g_{k,1,\infty})} \\
&= \mathcal{O}(k^{1/2}).
\end{aligned}$$

If we define Ω_∞ to be the unique solution to

$$\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}(\Omega_\infty) = -\Psi_{\perp,2,\infty}, \tag{5.82}$$

that is

$$\Omega_\infty = G_\infty\left(-\Psi_{\perp,2,\infty}\right),$$

we have

$$\begin{aligned}
&\|\Omega_s - \Omega_\infty\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} \qquad (5.83)\\
\leq\ & \|\Omega_s - G_s\left(-\Psi_{\perp,2,\infty}\right)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} + \|G_s\left(-\Psi_{\perp,2,\infty}\right) - G_\infty\left(-\Psi_{\perp,2,\infty}\right)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty})} \\
=\ & \mathcal{O}(k^{1/2}).
\end{aligned}$$

In particular we get that Ω_s converges to Ω_∞ smoothly.

Formally, from equations 5.76 and 5.83, we also get equation 5.29:

$$\frac{\partial \widehat{\omega}_{k,2}(s(t))}{\partial t} + i\bar{\partial}_J\partial_J\left(Scal\left(\widehat{\omega}_{k,2}(s(t))\right)\right) = \mathcal{O}(k^{-3}), \tag{5.84}$$

for all $s \in [0, S]$.

We now define

$$H(\omega_{k,2}(s(t))) \tag{5.85}$$

$$= \quad H(\omega''_{k,1}(s(t))) - k^{-2}\left(\Psi_{\perp,2}(s) + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s))\right) + k^{-3}\mathcal{L}_{V_s}(\Omega_s).$$

Then with this definition we also obtain equation 5.28, that is, for all $s \in [0,S]$,

$$\begin{aligned} i\partial_{J_s}\partial_{J_s}\left(H(\omega_{k,2}(s(t))\right) &= i\partial_{J_s}\partial_{J_s}\left(H(\omega''_{k,1}(s(t)))\right) \\ &\quad -k^{-2}i\partial_{J_s}\partial_{J_s}\left(\Psi_{\perp,2}(s) + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s))\right) \\ &\quad +k^{-3}\mathcal{L}_{V_s}(\Omega_s) \\ &= rk^{-1}\left(\frac{\partial\omega''_{k,1}(s(t))}{\partial s} + \mathcal{L}_{V_s}\left(\omega''_{k,2}(s(t))\right)\right) + k^{-2}\frac{\partial\Omega_s}{\partial t} \\ &= rk^{-1}\left(\frac{\partial\left(\omega''_{k,1}(s(t)) + k^{-2}\Omega_s\right)}{\partial s} + \mathcal{L}_{V_s}\left(\omega''_{k,2}(s(t)) + k^{-2}\Omega_s\right)\right) \\ &= rk^{-1}\left(\frac{\partial\omega_{k,2}(s(t))}{\partial s} + \mathcal{L}_{V_s}\left(\omega_{k,2}(s(t))\right)\right). \end{aligned} \tag{5.86}$$

In a similar way, we define

$$\begin{aligned} & H(\omega_{k,2,\infty}) \\ = \quad & H(\omega''_{k,1,\infty}) + k^{-3}\mathcal{L}_{V_\infty}(\Omega_\infty), \end{aligned} \tag{5.87}$$

where

$$\omega_{k,2,\infty} = \omega''_{k,1,\infty} + k^{-2}i\partial_{J_s}\partial_{J_s}(\Omega_\infty)$$

To finish the proof of Proposition 5.7 it remains to prove estimate 5.31, that is, we must estimate the quantity

$$\|Scal\left(\omega_{k,2}(s(t))\right) + H(\omega_{k,2}(s(t)) - \left(Scal\left(\omega_{k,2,\infty}\right) + H(\omega_{k,2,\infty})\right)\|_{W_{4,p,,q,w_\varepsilon(s)}(g_{k,1,\infty})}.$$

Formally we have

$$\begin{aligned} & Scal\left(\omega_{k,2}(s(t))\right) + H(\omega_{k,2}(s(t)) \\ = \quad & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}Scal\left(\omega_\Sigma\right) + \sum_{l=3} k^{-l}\left(\Psi^{(2)}_{\Sigma,l}(s) + \Psi^{(2)}_{\Phi_h,l}(s) + \Psi^{(2)}_{\perp,l}(s))\right) \\ & Scal\left(\omega_{k,2,\infty}\right) + H(\omega_{k,2,\infty}) \\ = \quad & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1}Scal\left(\omega_\Sigma\right) + \sum_{l=3} k^{-l}\left(\Psi^{(2)}_{\Sigma,l,\infty} + \Psi^{(2)}_{\Phi_h,l,\infty} + \Psi_{\perp,l,\infty}\right), \end{aligned}$$

where $\Psi^{(2)}_{\Sigma,l}(s), \Psi^{(2)}_{\Phi_h,l}(s), \Psi^{(2)}_{\perp,l}(s))\Psi^{(2)}_{\Sigma,l,\infty}, \Psi^{(2)}_{\Phi_h,l,\infty}, \Psi^{(2)}_{\perp,l,\infty}$ are defined by these formulae, so that

$$\begin{aligned} & Scal(\omega_{k,2}(s(t)) + H(\omega_{k,2}(s(t)) - Scal(\omega_{k,2}(s(t)) - H(\omega_{k,2}(s(t)) \\ = \quad & \sum_{l=3} k^{-l}\left((\Psi^{(2)}_{\Sigma,l}(s) - \Psi^{(2)}_{\Sigma,l,\infty}) + \left(\Psi^{(2)}_{\Phi_h,l}(s) - \Psi^{(2)}_{\Phi_h,l,\infty}\right) + \left(\Psi^{(2)}_{\perp,l}(s)) - \Psi^{(2)}_{\perp,l,\infty}\right)\right). \end{aligned} \tag{5.88}$$

5.4. **The parabolic estimate.** We may at last derive the estimate 5.31. Note that

$$\begin{aligned} & Scal\left(\omega_{k,2,\infty}\right) - Scal\left(\omega_{k,2}(s(t))\right) \\ = \quad & Scal\left(\omega_{k,1,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\Theta_\infty + k^{-1}i\overline{\partial}_{J_\infty}\partial_{J_\infty}\Xi_\infty + k^{-2}i\overline{\partial}_{J_\infty}\partial_{J_\infty}\Omega_\infty)\right) \end{aligned}$$

$$-Scal\left(\omega_{k,1}(s) + i\overline{\partial}_{J_s}\partial_{J_s}((\Theta(s(t) - \Theta_\infty) + \Theta_\infty) + k^{-1}i\overline{\partial}_{J_s}\partial_{J_s}\left((\Xi(s(t)) - \Xi_\infty) + \Xi_\infty\right) + k^{-2}i\overline{\partial}_{J_s}\partial_{J_s}\left(\Omega(s(t) - \Omega_\infty) + \Omega_\infty\right)\right)$$

so by Lemma 2.7 (or more precisely its proof) we have

$$\|Scal\left(\omega_{k,2}(s(t))\right) - \left(Scal\left(\omega_{k,2,\infty}\right)\right)\|^2_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}$$

$$
\begin{aligned}
: \quad &= \left\|\sum_{l=1}^{\infty} k^{-l}\left((\underline{\Psi}^{(2)}_{\Sigma,l}(s)-\underline{\Psi}^{(2)}_{\Sigma,l,\infty})+\left(\underline{\Psi}^{(2)}_{\Phi_h,l}(s)-\underline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right)+\left(\underline{\Psi}^{(2)}_{\perp,l}(s))-\underline{\Psi}^{(2)}_{\perp,l,\infty}\right)\right)\right\|^2_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})} \\
&\leq \sum_{l=1}^{\infty} k^{-l}\left\|\left((\underline{\Psi}^{(2)}_{\Sigma,l}(s)-\underline{\Psi}^{(2)}_{\Sigma,l,\infty})+\left(\underline{\Psi}^{(2)}_{\Phi_h,l}(s)-\underline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right)+\left(\underline{\Psi}^{(2)}_{\perp,l}(s))-\underline{\Psi}^{(2)}_{\perp,l,\infty}\right)\right)\right\|^2_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})} \\
&\leq C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left|\left|\left|\partial_t^j\left(\omega_{k,1,\infty}-\omega_{k,1}(s)\right)\right|\right|\right|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})} \\
&\quad C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left|\left|\left|\partial_t^j\left(\left(i\overline{\partial}_{J_\infty}\partial_{J_\infty}-i\overline{\partial}_{J_s}\partial_{J_s}\right)(\Theta_\infty+\Pi_\infty+\Omega_\infty)\right)\right|\right|\right|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j a_s^{(1,0)}\right\|_{L^2_{4(p+1-j)}(g_\infty)}+\left\|\partial_s^j a_s^{(0,1)}\right\|_{L^2_{4(p+1-j)}(g_\infty)} \\
&\quad C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j\left((\Theta(s(t)-\Theta_\infty)\right)\right\|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})}+\left\|\partial_s^j\left(a_s^{1,0}\wedge a_s^{0,1}\right)\right\|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j\left(\Xi(s(t))-\Xi_\infty\right)\right\|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})}+\left\|\partial_s^j\left(\Omega(s(t)-\Omega_\infty)\right)\right\|^2_{L^2_{4(p+1-j)}(g_{k,2,\infty})} \\
&\leq C vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\left(\left|\partial_t^j\left(\omega_{k,1,\infty}-\omega_{k,1}(s)\right)\right|\right)\right\|^2_{C^{4(p+1-j)}(g_{k,2,\infty})} \\
&\quad +C vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j a_s^{(1,0)}\right\|^2_{C^{4(p-j)}(g_{k,2,\infty})}+\left\|\partial_s^j a_s^{(0,1)}\right\|^2_{C^{4(p+1-j)}(g_{k,2,\infty})} \\
&\quad +C vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j\left(a_s^{1,0}\wedge a_s^{0,1}\right)\right\|^2_{C^{4(p-j)}(g_{k,2,\infty})} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q-1}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j\left(\left(\widehat{\Psi}_{\Sigma,2}(s(t)-\widehat{\Psi}_{\Sigma,2,\infty}\right)\right)\right\|^2_{L^2_{4(p-j)}(g_{k,2,\infty})} \\
&\quad C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q-1}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j(i(\beta(s)-\beta_\infty)+\partial_s\left(G_s(i\beta_\infty)\right)-g_S(s)\cdot\Delta_{A_s}(\overline{\eta}_s))\right\|_{L^2_{4(p-j)}(g_\Sigma,h)} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q-1}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j(G_s(i\beta_\infty)-G_\infty(i\beta_\infty)+g_S(s)\cdot\overline{\eta}_s)\right\|_{L^2_{4(p-j)}(g_\Sigma,h)} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q-1}\int_T^{\infty}|w_\varepsilon(t)|^2\left\|\partial_s^j(-\left(\Psi_{\perp,2}(s)-\Psi_{\perp,2,\infty}\right)+\partial_s G_s\left(\Psi_{\perp,2,\infty}\right))\right\|_{L^2_{4(p-j)}(g_{k,2,\infty})} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\sum_{j=0}^{q-1}\int_T^{\infty}|w_\varepsilon(t)|\left\|\partial_s^j(G_s\left(-\Psi_{\perp,2,\infty}\right)-G_\infty\left(-\Psi_{\perp,2,\infty}\right))\right\|_{L^2_{4(p-j)}(g_{k,2,\infty})} \\
&\quad +C\sum_{l=1}^{\infty} k^{-l}\left\|\Theta_\infty\right\|_{L^2_{4p+2}(g_{k,2,\infty})}+\left\|G_0(i\beta_\infty)\right\|_{L^2_{4p+2}(g_\Sigma,h)}+\left\|G_0\left(\Psi_{\perp,2,\infty}\right)\right\|_{L^2_{4p+2}(g_{k,1,\infty})} \\
&= C\sum_{l=1}^{\infty} k^{-l+1/2}+C\sum_{l=1}^{\infty} k^{-l} vol(\mathbb{P}(E),g_{k,2,\infty})\left(\left\|\Theta_\infty\right\|_{C^{4p+2}(g_{k,2,\infty})}+\left\|G_0\left(\Psi_{\perp,2,\infty}\right)\right\|_{C^{4p+2}(g_{k,2,\infty})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\left\|G_0(i\beta_\infty)\right\|_{C^{4p+2}(g_\Sigma,h)}\\
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{j=0}^{q-1}\int_T^\infty |w_\varepsilon(t)|^2\left\|\partial_s^j\left(\left(\widehat{\Psi}_{\Sigma,2}(s(t)-\widehat{\Psi}_{\Sigma,2,\infty}\right)\right)\right\|^2_{C^{4(p-j)}(g_{k,2,\infty})}\\
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{j=0}^{q-1}\int_T^\infty \left\|\partial_s^j(-(\Psi_{\perp,2}(s)-\Psi_{\perp,2,\infty})+\partial_sG_s\left(\Psi_{\perp,2,\infty}\right))\right\|_{C^{4(p-j)}(g_{k,2,\infty})}\\
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{j=0}^{q-1}\int_T^\infty |w_\varepsilon(t)|^2\left\|\partial_s^j(G_s(i\beta_\infty)-G_\infty(i\beta_\infty)+g_S(s)\cdot\overline{\eta}_s)\right\|_{C^{4(p-j)}(g_\Sigma,h)}\\
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{j=0}^{q-1}\int_T^\infty |w_\varepsilon(t)|^2\left\|i(\beta(s)-\beta_\infty)+\partial_s\left(G_s(i\beta_\infty)-g_S(s)\cdot\Delta_{A_s}(\overline{\eta}_s)\right)\right\|_{C^{4(p-j)}(g_\Sigma,h)}\\
&+C\sum_{l=1}^{\infty}k^{-l}vol(\mathbb{P}(E),g_{k,2,\infty})\sum_{j=0}^{q-1}\int_T^\infty |w_\varepsilon(t)|\left\|\partial_s^j(G_s\left(-\Psi_{\perp,2,\infty}\right)-G_\infty\left(-\Psi_{\perp,2,\infty}\right))\right\|_{C^{4(p-j)}(g_{k,2,\infty})}\\
=\;&\mathcal{O}(k^{1/2}),
\end{aligned}
$$

where we have used the inequalities 5.38, 5.61, 5.83 and Lemma 4.16, and the fact that by Lemmas 4.14 and 4.20, the constant appearing above is independent of k. Note also that Lemmas 4.14 and 4.20 work just as well for the metrics $g_{k,2,\infty}$ as for $g_{k,2,\infty}$. Note also that by Lemma 4.13, we have

$$
\begin{aligned}
\left\|\underline{\Psi}^{(2)}_{\Sigma,l}(s)-\underline{\Psi}^{(2)}_{\Sigma,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\underline{\Psi}^{(2)}_{\Phi_h,l}(s)-\underline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\underline{\Psi}^{(2)}_{\perp,l}(s))-\underline{\Psi}^{(2)}_{\perp,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}).
\end{aligned}
$$

By construction, we have that

$$
\begin{aligned}
H(\omega_{k,2}(s(t))) \;=\;& k^{-1}(2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s}))-k^{-2}\left(\Psi_{\perp,2}(s)+\widehat{\Psi}_{\Sigma,2}(s)+\Psi_{\Phi_h,l}(s)\right)\\
&-rk^{-2}\Phi_h(i\Delta_{A_s^{EndE}}(\eta_s)+2\Delta_{\omega_\Sigma}\Theta(w)\cdot\Lambda_{\omega_\Sigma}F_{A_s})-2\Lambda_{\omega_\Sigma}F_{A_s}\circ\eta_s)\\
&-k^{-2}\left(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta(w)+\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_s))\right)\\
&+k^{-3}\mathcal{L}_{V_s}\left(\Omega_s\right)\\
&+2rk^{-1}\Phi_h(\Lambda_{\omega_\Sigma}F_{A_s})\sum_{i=2}^{\infty}(-1)^ik^{-i}\left(\Phi_h(\eta_s)\right)^i\\
&+rk^{-2}\Phi_h\left(\frac{\partial\eta_s}{\partial s}+2\Lambda_{\omega_\Sigma}F_{A_s}\circ\eta_s\right)\sum_{i=1}^{\infty}(-1)^ik^{-i}\left(\Phi_h(\eta_s)\right)^i,
\end{aligned}
$$

and also

$$
\begin{aligned}
H(\omega_{k,2,\infty}) \;=\;& k^{-1}(2r\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty}))-k^{-2}\left(\Psi_{\perp,2,\infty}+\widehat{\Psi}_{\Sigma,2,\infty}+\Psi_{\Phi_h,l,\infty}\right)\\
&-rk^{-2}\Phi_h(i\Delta_{A_\infty^{EndE}}(\eta_\infty)+2\Delta_{\omega_\Sigma}\Theta_\infty\cdot\Lambda_{\omega_\Sigma}F_{A_\infty})-2\Lambda_{\omega_\Sigma}F_{A_\infty}\circ\eta_\infty)\\
&-k^{-2}\left(\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta_\infty+\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}(\Omega_\infty))\right)\\
&+k^{-3}\mathcal{L}_{V_\infty}\left(\Omega_\infty\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2rk^{-1}\Phi_h(\Lambda_{\omega_\Sigma}F_{A_\infty})\sum_{i=2}^{\infty}(-1)^i k^{-i}\left(\Phi_h(\eta_\infty)\right)^i\\
&+rk^{-2}\Phi_h\left(2\Lambda_{\omega_\Sigma}F_{A_\infty}\circ\eta_\infty\right)\sum_{i=1}^{\infty}(-1)^i k^{-i}\left(\Phi_h(\eta_\infty)\right)^i
\end{aligned}
$$

Then in the same way, applying Lemmas 4.15, 4.10, and 5.5, as well as the inequalities 5.38, 5.61, 5.83, we obtain that

$$
\begin{aligned}
&\|H(\omega_{k,2}(s(t)) - H(\omega_{k,2,\infty})\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,2,\infty})}\\
: \quad = &\left\|\sum_{l=1} k^{-l}\left((\overline{\Psi}^{(2)}_{\Sigma,l}(s) - \overline{\Psi}^{(2)}_{\Sigma,l,\infty}) + \left(\overline{\Psi}^{(2)}_{\Phi_h,l}(s) - \overline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right) + \left(\overline{\Psi}^{(2)}_{\perp,l}(s)) - \overline{\Psi}^{(2)}_{\perp,l,\infty}\right)\right)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
\leq \quad &\sum_{l=1} k^{-l}\left\|(\overline{\Psi}^{(2)}_{\Sigma,l}(s) - \overline{\Psi}^{(2)}_{\Sigma,l,\infty}) + \left(\overline{\Psi}^{(2)}_{\Phi_h,l}(s) - \overline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right) + \left(\overline{\Psi}^{(2)}_{\perp,l}(s)) - \overline{\Psi}^{(2)}_{\perp,l,\infty}\right)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
= \quad &\mathcal{O}(k^{1/2}).
\end{aligned}
$$

We note that by Lemma 4.13 we also obtain

$$
\begin{aligned}
\left\|\overline{\Psi}^{(2)}_{\Sigma,l}(s) - \overline{\Psi}^{(2)}_{\Sigma,l,\infty}\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\overline{\Psi}^{(2)}_{\Phi_h,l}(s) - \overline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\overline{\Psi}^{(2)}_{\perp,l}(s)) - \overline{\Psi}^{(2)}_{\perp,l,\infty}\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}).
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\|Scal\,(\omega_{k,2}(s(t))) + H(\omega_{k,2}(s(t)) - (Scal\,(\omega_{k,2,\infty}) + H(\omega_{k,2,\infty}))\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}\\
= \quad &\left\|\sum_{l=3} k^{-l}\left((\Psi^{(2)}_{\Sigma,l}(s) - \Psi^{(2)}_{\Sigma,l,\infty}) + \left(\Psi^{(2)}_{\Phi_h,l}(s) - \Psi^{(2)}_{\Phi_h,l,\infty}\right) + \left(\Psi^{(2)}_{\perp,l}(s)) - \Psi^{(2)}_{\perp,l,\infty}\right)\right)\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
\leq \quad &\sum_{l=3}^{\infty} k^{-l}\left(\left\|\underline{\Psi}^{(2)}_{\Sigma,l}(s) - \underline{\Psi}^{(2)}_{\Sigma,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|\underline{\Psi}^{(2)}_{\Phi_h,l}(s) - \underline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\right)\\
&+\sum_{l=3}^{\infty} k^{-l}\left\|\underline{\Psi}^{(2)}_{\perp,l}(s)) - \underline{\Psi}^{(2)}_{\perp,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|\overline{\Psi}^{(2)}_{\Sigma,l}(s) - \overline{\Psi}^{(2)}_{\Sigma,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
&+\sum_{l=3} k^{-l} + \left\|\overline{\Psi}^{(2)}_{\Phi_h,l}(s) - \overline{\Psi}^{(2)}_{\Phi_h,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} + \left\|\overline{\Psi}^{(2)}_{\perp,l}(s)) - \overline{\Psi}^{(2)}_{\perp,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})}\\
= \quad &\mathcal{O}(k^{-5/2}),
\end{aligned}
$$

for each l, where we use the previous calculations, and also Equation 5.88. Finally, again by Lemma 4.13 we obtain that

$$
\begin{aligned}
\left\|\Psi^{(2)}_{\Sigma,l}(s) - \Psi^{(2)}_{\Sigma,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\Psi^{(2)}_{\Phi_h,l}(s) - \Psi^{(2)}_{\Phi_h,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}),\\
\left\|\Psi^{(2)}_{\perp,l}(s)) - \Psi^{(2)}_{\perp,l,\infty}\right\|_{W_{4,p,,q-1,w_\varepsilon(s)}(g_{k,1,\infty})} &\leq \mathcal{O}(k^{1/2}).
\end{aligned}
$$

□

5.5. **The proof of Theorem 5.1.** Now we spell out how to iterate the above process to obtain an approximate solution to Calabi flow for all orders.

Proof. The proof will be by induction on l. The results of the two preceding subsections give Theorem 5.1 for $l = 1$ and 2. We simply iterate the procedure of the procedure, of the last subsection used to go from $l = 1$ to $l = 2$, which applies almost unchanged.

Suppose the result is true for l, that is, suppose that we have functions

$$\Theta_{k,m}(s(t)), \Xi_{k,m}(s(t)), \text{ and } \Omega_{k,m}(s(t))$$

for every $1 \leq m \leq l-1$, as in Theorem 5.1, so that in particular we may assume the existence of the the metrics $\omega_{k,m+1}(s(t))$ and the functions $H(\omega_{k,m+1}(s(t))$ (and in particular the metric $\omega_{k,l}(s(t))$), as well as the fact that the functions defined by

$$\begin{aligned}
& Scal\,(\omega_{k,m+1}(s(t))) + H(\omega_{k,m+1}(s(t)) \\
= \;& Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\,(\omega_\Sigma) \\
+ \;& \sum_{M=m+2} k^{-M}(\Psi^{(m+1)}_{\Sigma,M}(s) + \Psi^{(m+1)}_{\Phi_h,M}(s) + \Psi^{(m+1)}_{\perp,l}(s))
\end{aligned} \tag{5.89}$$

$$\begin{aligned}
& Scal\,(\omega_{k,m+1,\infty}) + H(\omega_{k,m+1,\infty}) \\
= \;& Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\,(\omega_\Sigma) \\
+ \;& \sum_{M=m+2} k^{-M}(\Psi^{(m+1)}_{\Sigma,M,\infty} + \Psi^{(m+1)}_{\Phi_h,M,\infty} + \Psi^{(m+1)}_{\perp,M,\infty})
\end{aligned} \tag{5.90}$$

satisfy the bounds

$$\begin{aligned}
\left\|\Psi^{(m+1)}_{\Sigma,M}(s) - \Psi^{(m+1)}_{\Sigma,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\
\left\|\Psi^{(m+1)}_{\Phi_h,M}(s) - \Psi^{(m+1)}_{\Phi_h,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\
\left\|\Psi^{(m+1)}_{\perp,M}(s)) - \Psi^{(m+1)}_{\perp,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}),
\end{aligned} \tag{5.91}$$

as in the case of $l = 2$ for which these facts are precisely equations 5.27, 5.30, of Proposition 5.7 in the previous section. In particular we may assume this this is true for $m = l - 1$. We will show the existence of functions

$$\Theta_{k,l}(w(t)), \Xi_{k,l}(s(t)), \text{ and } \Omega_{k,l}(s(t)),$$

such that the metric $\omega_{k,l+1}(s(t))$ defined in Theorem 5.1 achieves the desired result.

First, we define paths of functions

$$\widetilde{\Theta}_{k,l}(w(t)), \text{and } \widetilde{\Omega}_{k,l}(s(t))$$

and a path of endomorphisms $\widetilde{\eta}_{s,l}$ as follows.

We define $\widetilde{\Theta}_{k,l}(w(t))$ to be the solution to the initial value problem

$$\begin{aligned}
\frac{\partial \widetilde{\Theta}_{k,l}(w(t))}{\partial w} + \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\widetilde{\Theta}_{k,l}(w(t)) &= -\left(\widehat{\Psi}^{(l)}_{\Sigma,l+1}(s) - \widehat{\Psi}^{(l)}_{\Sigma,l+1,\infty}\right) \\
\widetilde{\Theta}_{k,l}(0) &= -\Theta_{k,l,\infty},
\end{aligned} \tag{5.92}$$

where $\Theta_{k,l,\infty}$ solves the elliptic equation

$$\mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta_{k,l,\infty} = -\widehat{\Psi}^{(l)}_{\Sigma,l+1,\infty}. \tag{5.93}$$

This solution exists by the parabolic Sobolev bound on $\widehat{\Psi}^{(l)}_{\Sigma,l+1}(s) - \widehat{\Psi}^{(l)}_{\Sigma,l+1,\infty}$, which has been assumed, and the fact that by construction

$$\widehat{\Psi}^{(l)}_{\Sigma,l+1}(s) - \widehat{\Psi}^{(l)}_{\Sigma,l+1,\infty}, \widehat{\Psi}^{(l)}_{\Sigma,l+1,\infty}$$

are orthogonal to

$$\ker \mathfrak{D}_\Sigma \mathfrak{D}_\Sigma = \mathbb{R}.$$

Now we define $\Theta_{k,l}(w(t))$ by

$$\Theta_{k,l}(w(t)) = \widetilde{\Theta}_{k,l}(w(t)) + \Theta_{k,l,\infty}, \tag{5.94}$$

This in particular forces $\Theta_{k,l}(w(t))$ to solve the intial value problem

$$\begin{aligned} \frac{\partial \Theta_{k,l}(w(t))}{\partial w} + \mathfrak{D}^*_{\omega_\Sigma}\mathfrak{D}_{\omega_\Sigma}\Theta_{k,l}(w(t)) &= -\widehat{\Psi}^{(l)}_{\Sigma,l+1}(s) \\ \Theta_{k,l}(0) &= 0. \end{aligned} \tag{5.95}$$

The parabolic Sobolev theory will imply that

$$\Theta_{k,l}(w(t)) \to \Theta_{k,l,\infty}$$

in C^∞, just as in the previous subsection, and that there is a bound

$$\|\Theta_{k,l}(w(t)) - \Theta_{k,l,\infty}\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = \left\|\widetilde{\Theta}_{k,l}(w(t))\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = \mathcal{O}(k^{1/2}). \tag{5.96}$$

Next we define $\widetilde{\eta}_{s,l}$ to be the solution to the initial value problem

$$\begin{aligned} \frac{\partial \widetilde{\eta}_{s,l}}{\partial s} + \Delta_{A_s^{EndE}}(\widetilde{\eta}_{s,l}) &= -i(\beta_l(s) - \beta_{l,\infty}) + \partial_s\left(G_s(i\beta_{l,\infty})\right) - g_S(s)\cdot\Delta_{A_s}(\overline{\eta}_{sl}) \\ \widetilde{\eta}_{0,l} &= G_0(i\beta_{l,\infty}), \end{aligned} \tag{5.97}$$

where if

$$\Phi_h\left(2r\alpha_l(s)\right) = \Psi^{(l)}_{\Phi_h,l+1}(s), \tag{5.98}$$

we have

$$-\Delta_{\omega_\Sigma}\Theta_{k,l}(w(t))\cdot\Lambda_{\omega_\Sigma}F_{A_s} + \alpha_l(s) = \beta_l(s) + \sum_j c_{j,l}(s) Id_{Q_j},$$

where $\sum_i c_{j,l}(s) Id_{Q_j}$ is the projection onto $\ker \Delta_{A_\infty^{EndE}} = \mathbb{C}^l$, $\overline{\eta}_{s,l}$ is defined to be the solution the system of ordinary differential equations

$$\begin{aligned} \frac{d\overline{\eta}_{s,l}}{dt} &= -i\sum_j c_{j,l}(s) Id_{Q_j} \\ \overline{\eta}_{0,l} &= 0, \end{aligned}$$

for all time, and $g_S(s)$ is a cut-off function which vanishes on $[2S,\infty)$ and $g_S(s)\equiv 1$ on $[0,2S]$.

Note that

$$\beta_l(s) \perp \ker \Delta_{A_s^{EndE}}$$

for each s with respect to the metric, and $G_s(i\beta_{l,\infty})$ is the Green's operator for $\Delta_{A_s^{EndE}}$ applied to $i\beta_{l,\infty}$, which is defined to be the part of

$$-\Delta_{\omega_\Sigma}\Theta_{k,l,\infty}\cdot\Lambda_{\omega_\Sigma}F_{A_\infty} + \alpha_{l,\infty}$$

orthogonal to $\ker \Delta_{A_s^{EndE}}$, which therefore enjoys the property

$$i\beta_{l,\infty} \perp \ker \Delta_{A_s^{EndE}},$$

for all s, since $\mathbb{C}\cdot Id_E = \Delta_{A_s^{EndE}} \subset \ker \Delta_{A_\infty^{EndE}}$.

Just as in Step 2 of the previous subsection, we will be able to use all of this information to conclude that the right hand side of the above equation lies in the appropriate parabolic Sobolev space, and is orthogonal to $\ker \Delta_{A_\infty^{EndE}}$ (and therefore to $\Delta_{A_s^{EndE}}$) for all sufficiently large s, so that the solution therefore exists for all time, and satisfies the corresponding parabolic estimate. We may therefore define

$$\eta_{s,l} = \widetilde{\eta}_{s,l} + G_s(-i\beta_{l,\infty}) + g_S(s)\cdot\overline{\eta}_{s,l}, \tag{5.99}$$

so that by construction

$$\begin{aligned}\frac{\partial \eta_{s,l}}{\partial s} + \Delta_{A_s^{EndE}}(\eta_{s,l}) &= -i\beta_l(s) + \frac{\partial}{\partial s}\left(g_S(s)\cdot\overline{\eta}_{s,l}\right), \\ \eta_{0,l} &= 0,\end{aligned} \tag{5.100}$$

Here again we note that the right hand side of this equation is equal to $\Delta_{\omega_\Sigma}\Theta_{k,l,\infty}\cdot\Lambda_{\omega_\Sigma}F_{A_\infty} + \alpha_{l,\infty}$ for $s\in[0,S]$.

We use this solution to define the function

$$\Xi_{k,l}(s(t)) = -\sum_{i=1}^{\infty} k^{-(i-1)}\left(\Phi_h\left(i\eta_{s,l}\right)\right)^i, \tag{5.101}$$

so that if we define

$$h_{\eta_{s,l}} = h + k^{-l}h\cdot\eta_{s,l}, \tag{5.102}$$

we have

$$\omega_k(h_{\eta_{s,l}}, J_s) = \omega_k(h, J_s) + k^{-l}i\overline{\partial}_{J_s}\partial_{J_s}\left(\Xi_{k,l}(s(t))\right). \tag{5.103}$$

By the parabolic theory, η_s will converge smoothly, and therefore $\Xi_{k,l}(s(t))$ also converges smoothly to some smooth function

$$\Xi_{k,l,\infty}$$

and again the parabolic theory implies a bound of the form

$$\|\Xi_{k,l}(s(t)) - \Xi_{k,l,\infty}\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = \mathcal{O}(k^{1/2}). \tag{5.104}$$

Finally we define $\widetilde{\Omega}_{k,l}(s(t))$ to be the solution to the initial value problem

$$\begin{aligned}&\frac{\partial}{\partial t}\widetilde{\Omega}_{k,l}(s(t)) + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\widetilde{\Omega}_{k,l}(s(t))) \\ &= -\left(\Psi^{(l)}_{\perp,l+1}(s) - \Psi^{(l)}_{\perp,l+1,\infty}\right) + \partial_t G_s\left(\Psi^{(l)}_{\perp,l+1,\infty}\right), \\ \widetilde{\Omega}_{k,l}(0) &= G_0(\Omega_{k,l,\infty}),\end{aligned} \tag{5.105}$$

where G_s is the Green's operator associated to $\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}$ and $\Omega_{k,l,\infty}$ is the solution to the elliptic equation

$$\mathfrak{D}^*_{(\mathcal{V}_\infty,h)}\mathfrak{D}_{(\mathcal{V}_\infty,h)}\left(\Omega_{k,l,\infty}\right) = -\Psi^{(l)}_{\perp,l+1,\infty}, \tag{5.106}$$

where again one proves exactly as in Step 3 of the previous subsection that the right hand side is the Sobolev space and orthogonal with respect to $g_{k,1,\infty}$ to $\ker\mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}$ for all s, and so the solution exists for all time. Now we may define

$$\Omega_{k,l}(s(t)) = \widetilde{\Omega}_{k,l}(s(t)) + G_s\left(-\Psi^{(l)}_{\perp,l+1,\infty}\right), \tag{5.107}$$

which therefore solves the the initial value equation

$$\frac{\partial}{\partial t}\Omega_{k,l}(s(t)) + \mathfrak{D}^*_{(\mathcal{V}_s,h)}\mathfrak{D}_{(\mathcal{V}_s,h)}(\Omega_{k,l}(s(t))) = -\Psi_{\perp,l+1}(s), \tag{5.108}$$

$$\Omega_{k,l}(0) \quad = \quad 0.$$

This solution will converge smoothly to $\Omega_{k,l,\infty}$, and the parabolic theory will imply the Sobolev bound

$$\|\Omega_{k,l}(s(t)) - \Omega_{k,l,\infty}\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} = \mathcal{O}(k^{1/2}).$$

Now we may define the one parameter family metrics

$$\begin{aligned}(5.109) \qquad & \omega_{k,l+1}(s(t)) \\ = \quad & \omega_{k,l}(s(t)) + k^{-(l-1)} i\bar{\partial}_{J_s}\partial_{J_s}\left(\Theta_{k,l}(w(t))\right) + k^{-l} i\bar{\partial}_{J_s}\partial_{J_s}\left(\Xi_{k,l}(s(t))\right) + k^{-(l+1)} i\bar{\partial}_{J_s}\partial_{J_s}\left(\Omega_{k,l}(s(t))\right).\end{aligned}$$

All the calculations of the preceeding subsection apply verbatim to this construction, the only difference being the correction potentials eliminate the terms

$$\Psi^{(l)}_{\Sigma,l+1}(s), \Psi^{(l)}_{\Phi_h,l+1}(s), \Psi^{(l)}_{\perp,l+1}(s),$$

because we have increased l, so that we formally obtain

$$\begin{aligned}(5.110) \qquad & Scal\left(\omega_{k,l+1}(s(t))\right) + H(\omega_{k,l}(s(t)) \\ = \quad & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\left(\omega_\Sigma\right) \\ + \quad & \sum_{M=l+2} k^{-M}(\Psi^{(l+1)}_{\Sigma,M}(s) + \Psi^{(l+1)}_{\Phi_h,M}(s) + \Psi^{(l+1)}_{\perp,M}(s)),\end{aligned}$$

where the function $H(\omega_{k,l+1}(s(t))$ is constructed from $H(\omega_{k,l}(s(t))$ in a completely analogous manner to the way $H(\omega_{k,2}(s(t))$ was constructed from $H(\omega_{k,1}(s(t))$, and by definition we have for $s \in [0, S]$

$$(5.111) \qquad rk^{-1}\left(\frac{\partial \omega_{k,l+1}(s)}{\partial s} + \mathcal{L}_{V_s}\omega_{k,l+1}(s)\right) = i\bar{\partial}_{J_s}\partial_{J_s} H(\omega_{k,l+1}(s)),$$

and precisely the same arguments apply to the solutions of the elliptic equations so that we may define $\omega_{k,l+1,\infty}$ analogously, and we have formally

$$\begin{aligned}(5.112) \qquad & Scal\left(\omega_{k,l+1,\infty}\right) + H(\omega_{k,l+1,\infty}) \\ = \quad & Scal\left(\omega_{FS}(\mathbb{P}^{r-1})\right) + k^{-1} Scal\left(\omega_\Sigma\right) \\ + \quad & \sum_{M=l+2} k^{-M}(\Psi^{(l+1)}_{\Sigma,M,\infty} + \Psi^{(l+1)}_{\Phi_h,M,\infty} + \Psi^{(l+1)}_{\perp,M,\infty}(s)),\end{aligned}$$

and by construction $\omega_{k,l+1}(s(t)) \to \omega_{k,l+1,\infty}$.

Moreover we may write

$$H(\omega''_{k,l+1,\infty}) = 2rk^{-1}\Phi_h\left(\Lambda_{\omega_\Sigma} F^\circ_{A_\infty} - \bigoplus_j \left(\sum_{q=2}^{l+1} \frac{k^{q-1}}{2} c_{j,q,\infty}\right) Id_{Q_j}\right) + \mathcal{O}(k^{-2}).$$

Precisely the same calculations used previously, using the Sobolev bounds on

$$\Theta_{k,l}(w(t)) - \Theta_{k,l,\infty}, \Xi_{k,l}(s(t)) - \Xi_{k,l,\infty}, \text{ and } \Omega_{k,l}(s(t)) - \Omega_{k,l,\infty}$$

apply to give the Sobolev bounds

$$(5.113) \qquad \begin{aligned}\left\|\Psi^{(l+1)}_{\Sigma,M}(s) - \Psi^{(l+1)}_{\Sigma,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\ \left\|\Psi^{(l+1)}_{\Phi_h,M}(s) - \Psi^{(l+1)}_{\Phi_h,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2}) \\ \left\|\Psi^{(l+1)}_{\perp,M}(s)) - \Psi^{(l+1)}_{\perp,M,\infty}\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})} &= \mathcal{O}(k^{1/2})\end{aligned}$$

for all $M \geq l+2$, and therefore

$$\begin{aligned} & \|Scal\,(\omega_{k,l+1}(s(t))) + H(\omega_{k,l+1}(s(t)) - (Scal\,(\omega_{k,l+1,\infty}) + H(\omega_{k,l+1,\infty}))\|_{W_{4,p,,q,w_\varepsilon(s)}(g_{k,1,\infty})} \\ = \; & \mathcal{O}(k^{-(l+1+1/2}). \end{aligned}$$

Finally we remark that Equation 5.11 also follows since the only facts we have used that involved the metric $g_{k,l+1,\infty}$ are the parabolic estimates (which are valid for any metric), the Sobolev embedding theorem Lemma 5.8 in [F], which is actually stated explicitly for $g_{k,l+1,\infty}$ for any l, and Lemma 4.15, which one can easily check by examining the proof, as well as the results used therein, that this lemma is equally valid for the metrics $g_{k,l+1,\infty}$ for each l, and so the equation follows with exactly the same proof. We have therefore proven that the result holds for $l+1$ and completing the proof of the theorem, by induction. □

6. Inverse function theorem argument

6.1. **Strategy of the proof.** We wish to find a path $\widehat{\omega}_t$ of smooth metrics solving the Calabi flow equation

$$\frac{\partial \widehat{\omega}_t}{\partial t} + i\overline{\partial}_J \partial_J Scal(\widehat{\omega}_t) = 0,$$

For the one parameter family (depending on S) of paths of metrics $\widehat{\omega}^S_{k,l}(s(t))$ provided by Theorem 5.1, where $s = rt/k$ we have that

$$\frac{\partial \widehat{\omega}^S_{k,l}(s(t))}{\partial t} + i\overline{\partial}_J \partial_J Scal(\widehat{\omega}^S_{k,l}(s(t))) = i\overline{\partial}_J \partial_J \widehat{\sigma}^S_{k,l}(s),$$

where $\widehat{\sigma}^S_{k,l}(s) = \widetilde{g}^*_s(\sigma^S_{k,l}(s))$ and $\sigma^S_{k,l}(s)$ satisfies a uniform estimate of the form

$$\left|\sigma^S_{k,l}(s)\right| \leq Ck^{-(l+1)},$$

on the interval $[0, S]$ (where we may take S to be as large as we like by modifying the cut-off function which introduced this parameter).

Note that since $\omega_k(h_t, J)$ and $\omega_k(h, J)$ are cohomologous for all t, by the $\overline{\partial}\partial$ lemma and the statement of Theorem 5.1, we may write $\widehat{\omega}^S_{k,l}(s(t)) = \omega_k(h, J) + i\overline{\partial}_J \partial_J \widehat{\varphi}^S_{k,l}(s)$ for a family of smooth functions $\widehat{\varphi}_{k,l}(t)$, this is equivalent to an equation of the form

$$\frac{\partial \widehat{\varphi}^S_{k,l}(t)}{\partial t} + Scal(\widehat{\omega}^S_{k,l}(t)) = \widehat{\sigma}^S_{k,l}(t)$$

By construction

$$\frac{\partial \widehat{\omega}_{k,l}(s(t))}{\partial t} = i\overline{\partial}_J \partial_J H\left(\widehat{\omega}_{k,l}(s(t))\right),$$

which implies

$$i\overline{\partial}_J \partial_J \left(\frac{\partial}{\partial t}\widehat{\varphi}_{k,l}(s(t))\right) = i\overline{\partial}_J \partial_J H\left(\widehat{\omega}_{k,l}(s(t))\right),$$

so by possibly adding a constant to $\widehat{\varphi}_{k,l}(s(t))$, we may assume

$$\frac{\partial}{\partial t}\widehat{\varphi}_{k,l}(s(t)) = H\left(\widehat{\omega}_{k,l}(s(t))\right), \tag{6.1}$$

or

$$rk^{-1}\left(\frac{\partial \varphi_{k,l}(s)}{\partial s} + \mathcal{L}_{V_s}\left(\varphi_{k,l}(s)\right)\right) = H\left(\omega_{k,l}(s)\right) \tag{6.2}$$

and therefore we may write

$$Scal\left(\omega_{k,l}(s(t))\right) + H\left(\omega_{k,l}(s(t))\right) = \sigma_{k,l}(s).$$

In precisely the same way we may write

$$Scal\left(\omega_{k,l,\infty}\right)+H\left(\omega_{k,l,\infty}\right)=\sigma_{k,l,\infty},$$

where $\sigma_{k,l}(s)$ converges smoothly to a function $\sigma_{k,l,\infty}$. By Theorem 5.1 this implies a parabolic Sobolev bound:

$$\|\sigma_{k,l}(s)-\sigma_{k,l,\infty}\|_{W_{4,p,,q,w_\varepsilon(s)}(g_{k,l,\infty})}\leq Ck^{-(l+1/2)}.$$

If we can find a path of smooth functions $\phi(s(t))$ and a smooth function ϕ_∞ such that such that

(6.3)

$$Scal\left(\omega_{k,l}(s(t))+i\overline{\partial}_{J_s}\partial_{J_s}(\phi(s(t))+\phi_\infty)\right)+H\left(\omega_{k,l}(s(t))\right)+rk^{-1}\left(\frac{\partial}{\partial s}(\phi(s(t)))+\mathcal{L}_{V_s}\left(\phi(s(t))+\phi_\infty\right)\right)=0,$$

where V_s is the infinitesimal generator of the path of diffemorphisms $\widetilde{g}_s$, then

$$\begin{aligned}\widehat{\omega}(s(t)) &= \widetilde{g}_s^*\left(\omega_{k,l}(s(t))+i\overline{\partial}_{J_s}\partial_{J_s}(\phi(s(t))+\phi_\infty)\right)\\ &= \widehat{\omega}_{k,l}(s(t))+i\overline{\partial}_J\partial_J(\widehat{\phi}(s(t))+\widehat{\phi}_\infty)\\ &= \omega_k(h,J)+i\overline{\partial}_J\partial_J\left(\widehat{\varphi}_{k,l}(s(t))+\widehat{\phi}(s(t))+\widehat{\phi}_\infty\right),\end{aligned}\tag{6.4}$$

solves Calabi flow:

$$\begin{aligned}&Scal\left(\omega_k(h,J)+i\overline{\partial}_J\partial_J\left(\widehat{\varphi}_{k,l}(s(t))+\widehat{\phi}(s(t))+\widehat{\phi}_\infty\right)\right)+\frac{\partial}{\partial t}\left(\widehat{\varphi}_{k,l}(s(t))+\widehat{\phi}(s(t))+\widehat{\phi}_\infty\right)\\ =\; & Scal\left(\widehat{\omega}(s(t))\right)+\frac{\partial}{\partial t}\left(\widehat{\varphi}_{k,l}(s(t))+\widehat{\phi}(s(t))\right)\\ =\; & Scal\left(\widehat{\omega}(s(t))\right)+H\left(\widehat{\omega}_{k,l}(s(t))\right)+\frac{\partial}{\partial t}\left(\widehat{\phi}(s(t))+\widehat{\phi}_\infty\right)\\ =\; & 0.\end{aligned}$$

The idea then is to perturb the approximate solution $\varphi_{k,l}(s)$ to a genuine solution by adding a potential of the form $\phi(s(t))+\widehat{\phi}_\infty$. We will do this via an implicit function theorem argument. For $\phi(s(t))\in W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})$, we have by definition that $\phi(s(t))\to 0$ as $s\to 0$ in $L^2_{4(p+1)}(g_{k,l,\infty})$, so that for any $\phi_\infty\in L^2_{4(p+1)}(g_{k,l,\infty})$,

$$\phi(s(t))+\phi_\infty\to\phi_\infty$$

in $L^2_{4(p+1)}(g_{k,l,\infty})$. Then we may consider the Calabi maps

$$\mathcal{C}_{k,l}:W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,1,\infty})\to W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})\tag{6.5}$$

$$(\phi(s(t)),\phi_\infty)\mapsto\left(\tau_{k,l}\left(\phi(s(t)),\phi_\infty\right),\kappa_{k,l}(\phi_\infty)\right),\tag{6.6}$$

where

$$\begin{aligned}&\tau_{k,l}\left(\phi(s(t)),\phi_\infty\right)\qquad (6.7)\\ =\;& Scal\left(\omega_{k,l}(s(t))+i\overline{\partial}_{J_s}\partial_{J_s}\left(\phi(s(t))+\phi_\infty\right)\right)+H\left(\omega_{k,l}(s(t))\right)+rk^{-1}\left(\frac{\partial}{\partial s}\phi(s(t))+\mathcal{L}_{V_s}\left(\phi(s(t))+\phi_\infty\right)\right)\\ &-\left(Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)+H(\omega_{k,l,\infty}\right)+rk^{-1}\mathcal{L}_{V_\infty}\phi_\infty),\end{aligned}$$

and

$$\kappa_{k,l}(\phi_\infty)=Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)+H(\omega_{k,l,\infty})+rk^{-1}\mathcal{L}_{V_\infty}\phi_\infty.\tag{6.8}$$

and where $\varepsilon>1/2$.

We will now slightly rewrite these operators in a more familiar form. Recalling that

$$V_\infty\;=\;X_{-i\Lambda_{\omega_\Sigma}F_{A_\infty}}=\nabla_{g_{k,1,\infty}}\Phi_h\left(-\Lambda_{\omega_\Sigma}F_{A_\infty}\right),$$

$$X_{\Lambda_{\omega_\Sigma} F_{A_\infty}} = J_\infty \nabla_{g_{k,1,\infty}} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right) = J_\infty V_\infty,$$

so that $\Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right)$ is in particular a Hamiltonian for the Hamiltonian vector field $X_{\Lambda_{\omega_\Sigma} F_{A_\infty}}$ with respect to $\omega_{k,1,\infty}$.

By definition we have that

$$\omega_{k,l,\infty} = \omega_{k,1,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} (\varphi_{k,l,\infty} - \varphi_{k,1,\infty}),$$

so that by Lemma 2.4 and the limit of equation 6.1 as $s \to \infty$, for $l \geq 2$ we may write:

$$\begin{aligned} rk^{-1} H_{J_\infty V_\infty} \left(\omega_{k,l,\infty}\right) &= rk^{-1} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right) - \frac{rk^{-1}}{2} \mathcal{L}_{\nabla_{g_{k,1,\infty}} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right)} (\varphi_{k,l,\infty} - \varphi_{k,1,\infty}) \\ &= rk^{-1} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right) - \frac{1}{2} H(\omega_{k,l,\infty}) + rk^{-1} \Phi_h \left(\Lambda_{\omega_\Sigma} F_{A_\infty}\right) \\ &= -\frac{1}{2} H(\omega_{k,l,\infty}), \end{aligned}$$

where $H_{J_\infty V_\infty}(\omega_{k,l,\infty})$ is a Hamiltonian function for $X_{\Lambda_{\omega_\Sigma} F_{A_\infty}} = J_\infty V_\infty$ with respect to metric $\omega_{k,l,\infty}$. In other words the function $H(\omega_{k,l,\infty})$ is in fact -2 times this Hamiltonian function.

We therefore obtain

$$\begin{aligned} \kappa_{k,l}(\phi_\infty) &= Scal\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right) - 2rk^{-1} \left(H_{JV_\infty}(\omega_{k,l,\infty}) - \frac{1}{2} \mathcal{L}_{\nabla_{g_{k,1,\infty}} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right)} (\phi_\infty)\right) \\ &= Scal\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right) - \left(\Phi_h \left(\Lambda_{\omega_\Sigma} F_{A_\infty}\right) - \frac{1}{2} \mathcal{L}_{\nabla_{g_{k,1,\infty}} \Phi_h \left(-\Lambda_{\omega_\Sigma} F_{A_\infty}\right)} (\varphi_{k,l,\infty} - \varphi_{k,1,\infty} + \phi_\infty)\right) \\ &= Scal\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right) - 2rk^{-1} \left(H_{J_\infty V_\infty}\left(\omega_{k,1,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} (\varphi_{k,l,\infty} - \varphi_{k,1,\infty} + \phi_\infty)\right)\right) \\ &= Scal\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right) - 2rk^{-1} \left(H_{J_\infty V_\infty}\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right)\right), \end{aligned}$$

and also

$$\begin{aligned} &\tau_{k,l}\left(\phi(s(t)), \phi_\infty\right) \\ = \ & Scal\left(\omega_{k,l}(s(t)) + i\overline{\partial}_{J_s} \partial_{J_s} \left(\phi(s(t)) + \phi_\infty\right)\right) + H\left(\omega_{k,l}(s(t))\right) + rk^{-1} \left(\frac{\partial}{\partial s} \phi(s(t)) + \mathcal{L}_{V_s} \left(\phi(s(t)) + \phi_\infty\right)\right) \\ &- \left(Scal\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right) - 2rk^{-1} H_{J_\infty V_\infty}\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty} \partial_{J_\infty} \phi_\infty\right)\right). \end{aligned}$$

Notice that the operator $\kappa_{k,l}$ is precisely the extremal metric operator (see equation 2.5) for the vector field

$$-2rk^{-1} J_\infty V_\infty = 2rk^{-1} J_\infty \nabla_{g_{k,1,\infty}} \Phi_h (\Lambda_{\omega_\Sigma} F_{A_\infty}).$$

These maps are well-defined and differentiable on the space $W^0_{4,p+1,q,w_\varepsilon(t)}\left(g_{k,l,\infty}\right) \times L^2_{4(p+1)}\left(g_{k,l,\infty}\right)$ for all sufficiently large p by Lemma 2.7. The equation

$$\mathcal{C}_{k,l}\left(\left(\phi(s(t)), \phi_\infty\right)\right) = 0 \tag{6.9}$$

implies that

$$\widehat{\omega}(t) = \widehat{\omega}_{k,l}(s(t)) + i\overline{\partial}_J \partial_J (\widehat{\phi}(s(t) + \widehat{\phi}_\infty)$$

solves Calabi flow, by the above discussion, assuming we can actually take $\phi(s(t)) + \widehat{\phi}_\infty$ to be smooth.

By the previous discussion, there is also a pointwise uniform bound

$$|\mathcal{C}_{k,l}(0)| \leq Ck^{-(k+1)}.$$

In fact, by Theorem 5.1

$$\|C_{k,l}(0)\|_{W_{4,p,q,w_\varepsilon(t)}\left(g_{k,l,\infty}\right) \times L^2_{4p}\left(g_{k,l,\infty}\right)} \tag{6.10}$$

$$
\begin{aligned}
&= \|\sigma_{k,l}(s) - \sigma_{k,l,\infty}\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty}) \times L^2_{4p}(g_{k,1,\infty})} \\
&\leq Ck^{-(l+1/2)},
\end{aligned}
$$

for all p,q, and ε.

We would like to use a quantitative version of the inverse function theorem to find an exact solution to equation 6.9 and therefore to Calabi flow. Unfortunately, the linearisation of the scalar curvature involves the Lichnerowicz operators of the metrics $\omega_{k,l}(s(t))$ and $\omega_{k,l,\infty}$ (see Lemmas 2.5 and 2.7 and formula 6.21 below) which have kernels isomorphic to $\mathbb{R}$ and $\mathbb{R}^{m+1}$ by Lemmas 4.3 and 4.8. Because of this, we will not be able to solve Equation 6.9. Rather we will solve a modified version of this equation for an entire one parameter family of operators $\mathcal{C}^S_{k,l}$ (to be defined below) for which a solution to

$$
\mathcal{C}^S_{k,l}\left(\phi(s(t)), \phi_\infty\right) = 0, \tag{6.11}
$$

will give a solution to Equation 6.3 up to time S. Since we will solve this equation for every time S, we will therefore obtain a solution to Equation 6.3 and therefore to Calabi flow on the original manifold $(\mathbb{P}(\mathcal{E}), J)$ for all time.

Note that this problem is already present in Brönnle's solution to the (elliptic) extremal metric problem, that is the equation

$$
\kappa_{k,l}(\phi_\infty) = 0.
$$

To find the extremal metric on $\mathbb{P}(\mathcal{E}_\infty)$ Brönnle modifies the vector field $J_\infty V_\infty$ by a vertical vector field induced by a block-diagonal endomorphism, the Hamiltonian of which kills the orthogonal projection onto $\ker \mathfrak{D}^*_{(\omega_{k,1}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,1}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})}$, allowing him to carry out the perturbation. We will follow his method exactly for the component the second (non-time dependent) component $\kappa_{k,l}$ of our map (since this is exactly the same as Brönnle's map). The time dependent component $\tau_{k,l}$ is slightly trickier to deal with. We need to redefine it so that it accomplishes four things at once. First of all we need it to get rid of all the kernels involved. We will accomplish this for $\ker \mathfrak{D}^*_{(\omega_{k,1}(J_s,h))}\mathfrak{D}_{(\omega_{k,1}(J_s,h))}|_{C^\infty(X,\mathbb{R})}$ by adding a term given by the projection onto this component. Since this is constant for all s, this term will be zero after we take $i\overline{\partial}_{J_s}\partial_{J_s}$. For $\ker \mathfrak{D}^*_{(\omega_{k,1}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,1}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})}$ we accomplish this by defining the part of $\tau_{k,l}$ that is not time-dependent to be exactly the same as (the modified) $\kappa_{k,l}$, which also means that if $\kappa_{k,l}$ vanishes, $\tau_{k,l}$ is purely time dependent, as before. Therefore, secondly we must arrange that the time dependent part converges to $\kappa_{k,l}$ at an appropriate rate so that $\tau_{k,l}$ still gives a map between the parabolic Sobolev spaces. Thirdly, we need to know that the vanishing of the (modified) $\kappa_{k,l}$ and $\tau_{k,l}$ together (in other words of $\mathcal{C}^S_{k,l}$ for every $S \geq 0$) implies the existence of a solution to Equation 6.3. Finally, we need to know that the analogue of inequality 6.10 will still be satisfied. This last point will follow from any reasonable definition, and there is essentially only one way to acheive the first point (that is, by following Brönnle's method). To acheive points two and three simultaneously, we will use a cut-off function (which is where S appears) so that at infinity the time dependent part of $\tau_{k,l}$ converges to the time independent part, but up to time S it remains unchanged, so we also acheive the third point above (but only up to time S). This is why we are required to solve our equation for an entire one parameter family of operators, rather than a single operator as one does in the elliptic case.

We will start by setting up the definition of the operators $\mathcal{C}^S_{k,l}$. In order to deal with the kernels mentioned above, we will adopt Bronnle's framework from [B], to modify the non-time-dependent part $\kappa_{k,l}$ of $\mathcal{C}_{k,l}$, and then we will modify the time-dependent part $\tau_{k,l}$ of $\mathcal{C}_{k,l}$ accordingly.

Namely, recall that by Lemma 4.8 any element of $\ker \mathfrak{D}^*_{(\omega_{k,1}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,1}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})}$ will be equal to

$$\Phi_h(\oplus_j \theta_j Id_{Q_j}),$$

with $\theta_j \in i\mathbb{R}$, and these functions are precisely the (real-valued) Hamiltonians for the vector fields

$$X_\theta := X_{\oplus_j \theta_j Id_{Q_j}} = J_\infty \nabla_{g_{k,1,\infty}} \Phi_h(\oplus_j \theta_j Id_{Q_j}).$$

We will write λ_j° for the eigenvalues of $\Lambda_{\omega_\Sigma} F^\circ_{A_\infty}$, and

$$\vartheta_j := \lambda_j^\circ \theta_j,$$

and similarly

$$X_\vartheta := X_{\oplus_j \vartheta_j Id_{Q_j}}$$

Again, because

$$\omega_{k,l,\infty} = \omega_{k,1,\infty} + i\overline{\partial}_{j_\infty}\partial_{J_\infty}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty}\right),$$

where the difference $\varphi_{k,l,\infty} - \varphi_{k,1,\infty}$ satisfies by construction the property

$$\mathcal{L}_{X_\vartheta}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty}\right) = 0,$$

then by Lemma 2.4, if we set $\vartheta = (\vartheta_1, \cdots, \vartheta_m)$, the functions

$$\begin{aligned} H_{X_\vartheta}(\omega_{k,l,\infty}) &:= \Phi_h(\oplus_j \vartheta_j Id_{Q_i}) - \frac{1}{2} g_{k,1,\infty}\Big(\nabla_{g_{k,1,\infty}}\Phi_h(\oplus_j \vartheta_j Id_{Q_i}), \nabla_{g_{k,1,\infty}}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty}\right)\Big) \\ &= \mathcal{L}_{J_\infty X_\vartheta}\left(\varphi_{k,l,\infty}\right), \end{aligned} \tag{6.12}$$

where we have used that

$$\mathcal{L}_{J_\infty X_\vartheta}\left(\omega_{k,1,\infty}\right) = 2i\overline{\partial}_{J_\infty}\partial_{J_\infty}\Phi_h(\oplus_j \vartheta_j Id_{Q_i}).$$

Therefore the $H_{X_\vartheta}(\omega_{k,l,\infty})$ are Hamiltonians for X_ϑ with respect to the metric $\omega_{k,l,\infty}$, and so

$$\ker \mathfrak{D}^*_{(\omega_{k,l}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,l}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})} = \{H_{X_\vartheta}(\omega_{k,l,\infty}) \in (i\mathbb{R})^m\} \oplus \mathbb{R} \simeq \mathbb{R}^{m+1},$$

where the additional factor of $\mathbb{R}$ comes from the addition of a constant. More precisely, for any $\phi_\infty \in L^2_{4(p+1)}(g_{k,1,\infty})$, if we write $proj_{\ker \mathfrak{D}^*_{(\omega_{k,l}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,l}(J_\infty,h))}}$ for the L^2-orthogonal projection onto $\ker \mathfrak{D}^*_{(\omega_{k,1}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,1}(J_\infty,h))}|_{C^\infty(X,\mathbb{R})}$, we may find a pair $(\vartheta, R) \in \mathbb{R}^m \times \mathbb{R}$ such that

$$proj_{\ker \mathfrak{D}^*_{(\omega_{k,l}(J_\infty,h))}\mathfrak{D}_{(\omega_{k,l}(J_\infty,h))}}(\phi_\infty) = 2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty}) + R.$$

For any ϕ_∞ with

$$\mathcal{L}_{X_\vartheta}\left(\phi_\infty\right) = 0,$$

we may define

$$H_{X_\vartheta}(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(\phi\right))$$

in the same way. Note that the map

$$(\phi_\infty, \vartheta) \mapsto H_{X_\vartheta}(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(\phi\right))$$

is linear in both ϕ_∞ and ϑ, and so the linearisation of this map is given by

$$\begin{aligned} &\frac{d}{dw} H_{X_{w\vartheta}}(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\left(w\phi\right))|_{w=0} \\ = \;& \frac{d}{dw} w\Phi_h(\oplus_j \vartheta_j Id_{Q_i}) - w\frac{1}{2} g_{k,1,\infty}\Big(\nabla_{g_{k,1,\infty}}\Phi_h(\oplus_j \vartheta_j Id_{Q_i}), \nabla_{g_{k,1,\infty}}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty}\right)\Big)|_{w=0} \\ & - \frac{d}{dw} w^2 \frac{1}{2} g_{k,1,\infty}\Big(\nabla_{g_{k,1,\infty}}\Phi_h(\oplus_j \vartheta_j Id_{Q_i}), \nabla_{g_{k,1,\infty}}(\phi)\Big)|_{w=0} \\ = \;& H_{X_\vartheta}(\omega_{k,l,\infty}) = \mathcal{L}_{J_\infty X_{\vartheta(s)}}\left(\varphi_{k,l,\infty}\right). \end{aligned} \tag{6.13}$$

Now let $f_S(s)$ be a cut-off function which is 0 on the interval $[0,S]$ and 1 on the interval $[2S,\infty)$. We will also consider the path of vector fields

$$X^S_\vartheta(s) = f_S(s)\cdot J_\infty X_\vartheta = f_S(s) X_{i\Lambda_{\omega_\Sigma} F^\circ_{A_\infty}\cdot F_\theta},$$

where

$$F_\theta = \oplus_j \theta_j Id_{Q_j}.$$

Note that for $t \le S$ we have $X^S_\vartheta(s) = 0$, and for $t \ge 2S$, $X^S_\vartheta(s) = J_\infty X_\vartheta$.

In the same way, we will define

$$V^S_s := V_s(1-f_S) + f_s \nabla_{g_{\omega_{k,l}(s(t))}} H\left(\omega_{k,l}(s(t))\right),$$

so that $V^S_s = V_s$ for $s\in[0,S]$ and $V^S_s = \nabla_{g_{\omega_{k,l}(s(t))}} H\left(\omega_{k,l}(s(t))\right)$ for $[2S,\infty)$.

Note that we may write

$$\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty = \omega_{k,1,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\varphi_{k,l,\infty} - \varphi_{k,1,\infty} + \phi_\infty),$$

and so by Lemma 2.4 we have that

$$\begin{aligned}
& H_{J_\infty V_\infty}\left(\omega_{k,l,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right) \\
= \;& H_{J_\infty V_\infty}\left(\omega_{k,l,\infty}\right) - \frac{1}{2} g_{\omega_{k,l,\infty}}\left(\nabla_{g_{\omega_{k,l,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,l,\infty}\right), \nabla_{g_{\omega_{k,l,\infty}}}\left(\phi_\infty\right)\right) \\
= \;& H_{J_\infty V_\infty}\left(\omega_{k,1,\infty} + i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\varphi_{k,l,\infty} - \varphi_{k,1,\infty} + \phi_\infty)\right) \\
= \;& H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right) - \frac{1}{2} g_{\omega_{k,1}(s(t))}\left(\nabla_{g_{\omega_{k,1,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right), \nabla_{g_{\omega_{k,1,\infty}}}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty} + \phi_\infty\right)\right) \\
= \;& H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right) - \frac{1}{2} g_{\omega_{k,1}(s(t))}\left(\nabla_{g_{\omega_{k,1,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right), \nabla_{g_{\omega_{k,1,\infty}}}\left(\varphi_{k,l,\infty} - \varphi_{k,1,\infty}\right)\right) \\
& -\frac{1}{2} g_{\omega_{k,1}(s(t))}\left(H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right), \nabla_{g_{\omega_{k,l}(s(t))}}\left(\phi_\infty\right)\right) \\
= \;& H_{J_\infty V_\infty}\left(\omega_{k,l,\infty}\right) - \frac{1}{2} g_{\omega_{k,1}(s(t))}\left(\nabla_{g_{\omega_{k,1,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right), \nabla_{g_{\omega_{k,1,\infty}}}\left(\phi_\infty\right)\right)
\end{aligned}$$

so comparing the second and final lines above we obtain

$$\begin{aligned}
& \mathcal{L}_{\nabla_{g_{\omega_{k,l,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,l,\infty}\right)}\left(\phi_\infty\right) \\
= \;& \frac{1}{2} g_{\omega_{k,l}(s(t))}\left(\nabla_{g_{\omega_{k,l,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,l,\infty}\right), \nabla_{g_{\omega_{k,l}(s(t))}}\left(\phi_\infty\right)\right) \\
= \;& \frac{1}{2} g_{\omega_{k,1}(s(t))}\left(\nabla_{g_{\omega_{k,1,\infty}}} H_{J_\infty V_\infty}\left(\omega_{k,1,\infty}\right), \nabla_{g_{\omega_{k,1,\infty}}}\left(\phi_\infty\right)\right) \\
= \;& \mathcal{L}_{V_\infty}\left(\phi_\infty\right).
\end{aligned}$$

In particular, $\mathcal{L}_{V^S_s}\left(\phi(s(t) + \phi_\infty\right)$ converges smoothly to $\mathcal{L}_{V_\infty}\left(\phi_\infty\right)$.

Now we may define a one parameter family of parametrised Calabi operators

$$\mathcal{C}^S_{k,l} : W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty}) \times L^2_{4(p+1)}(g_{k,1,\infty}) \times \mathbb{R}^m \times \mathbb{R} \to W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty}) \times L^2_{4p}(g_{k,1,\infty}) \tag{6.14}$$

given by

$$(\phi(s(t)), \phi_\infty, \vartheta, R) \mapsto \left(\tau^\vartheta_{k,l}\left(\phi(s(t), \phi_\infty\right), \kappa^\vartheta_{k,l}(\phi_\infty, R)\right)$$

where

$$\begin{aligned}
& \tau^\vartheta_{k,l}\left(\phi(s(t)), \phi_\infty\right) \qquad (6.15) \\
= \;& Scal\left(\omega_{k,l}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s}\left(\phi(s(t)) + \phi_\infty\right)\right) + H\left(\omega_{k,l}(s(t))\right) + rk^{-1}\mathcal{L}_{X^S_\vartheta(s)}\left(\varphi_{k,l,\infty}\right)
\end{aligned}$$

$$
\begin{aligned}
&+rk^{-1}\left(\frac{\partial}{\partial s}\phi(s(t)) + \mathcal{L}_{V_s^S+X_\vartheta^S(s)}\left(\phi(s(t))+\phi_\infty\right)\right)\\
&-\left(Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right) - 2rk^{-1}H_{J_\infty V_\infty + X_\vartheta}\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right)\right),\\
= \; & Scal\left(\omega_{k,l}(s(t)) + i\overline{\partial}_{J_s}\partial_{J_s}\left(\phi(s(t))+\phi_\infty\right)\right) + H\left(\omega_{k,l}(s(t))\right) + rk^{-1}\mathcal{L}_{X_\vartheta^S(s)}\left(\varphi_{k,l,\infty}\right)\\
&+rk^{-1}\left(\frac{\partial}{\partial s}\phi(s(t)) + \mathcal{L}_{V_s^S+X_\vartheta^S(s)}\left(\phi(s(t))+\phi_\infty\right)\right)\\
&-\left(Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right) + H\left(\omega_{k,l,\infty}\right) + rk^{-1}\mathcal{L}_{J_\infty X_\vartheta(s)}\left(\varphi_{k,l,\infty}\right) + rk^{-1}\left(\mathcal{L}_{V_\infty+J_\infty X_\vartheta}\left(\phi_\infty\right)\right)\right),
\end{aligned}
$$

and where

$$
\begin{aligned}
\kappa_{k,l}^{\vartheta}(\phi_\infty,R)) &= \kappa_{k,l}(\phi_\infty) - 2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty)) - R\\
&= Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right) - 2rk^{-1}\left(H_{J_\infty V_\infty+X_\vartheta}(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}(\phi_\infty))\right) - R\\
&= Scal\left(\omega_{k,l,\infty}+i\overline{\partial}_{J_\infty}\partial_{J_\infty}\phi_\infty\right) + H\left(\omega_{k,l,\infty}\right)\\
&\quad +rk^{-1}\mathcal{L}_{J_\infty X_\vartheta(s)}\left(\varphi_{k,l,\infty}\right) + rk^{-1}\left(\mathcal{L}_{V_\infty+J_\infty X_\vartheta}\left(\phi_\infty\right)\right) - R
\end{aligned}
\tag{6.16}
$$

Clearly for every $S \geq 0$

$$
\mathcal{C}_{k,l}^S(0) = \mathcal{C}_{k,l}(0),
$$

so

$$
\left\|\mathcal{C}_{k,l}^S(0)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4p}(g_{k,1,\infty})} \leq Ck^{-(l+1/2)}. \tag{6.17}
$$

In order to obtain an exact solution to the equation

$$
\mathcal{C}_{k,l}^S\left((\phi(s(t)),\phi_\infty,\vartheta,R)\right) = 0, \tag{6.18}
$$

and therefore to equation 6.3 up to time S, we wish to apply the following theorem to the maps $\mathcal{C}_{k,l}^S$.

Theorem 6.1. *Let V and W be banch spaces, and $\mathcal{C}: U \to W$ a differentiable map whose derivative at 0 is an epimorphism, having right inverse $\mathcal{P}$: Then there is a neighbourhood $B_{\delta'}(0) \subset V$ on which the map $\mathcal{C} - d\mathcal{C}$ is Lipschitz with constant $\frac{2}{\|\mathcal{P}\|}$. Then if we set $\delta = \delta'\left(\frac{2}{\|\mathcal{P}\|}\right)$, for any $y \in B_\delta\left(\mathcal{C}(0)\right)$, there exists a unique $x \in B_{\delta'}(0)$ such that $\mathcal{C}(x) = y$.*

In the rest of this section, we complete the proof of Theorem 1.3 by establishing that the conditions of hold for the operators $\mathcal{C}_{k,l}^S$. In particular, we will need to establish control on both the linearisation, and the non-linear parts of these maps.

6.2. **A bounded inverse for the linearisation.** In this subsection we will prove the following proposition.

Proposition 6.2. *For $k >> 0$ and $l \geq 3$, the operator*

$$
(d\mathcal{C}_{k,l}^S)_0 : W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4(p+1)}(g_{k,l,\infty}) \times \mathbb{R}^{m+1} \to W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4p}(g_{k,l,\infty}) \tag{6.19}
$$

is a Banach space epimorphism with right inverse $\mathcal{P}_{k,l}$. There exists a constant C, such that for all $k >> 0$, and all $(\psi_t,\psi_\infty) \in W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4p}(g_{k,l,\infty})$, the inverse $\mathcal{P}_{k,l}$ satisfies the property

$$
\begin{aligned}
&\left\|\mathcal{P}_{k,l}\left((\psi_t,\psi_\infty)\right)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})\times\mathbb{R}^{m+1}}\\
\leq\; & Ck^3\left\|(\psi_t,\psi_\infty)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})}
\end{aligned}
\tag{6.20}
$$

$$= \quad Ck^3 \left(\|(\psi_t\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})} + ||\psi_\infty||_{L^2_{4p}(g_{k,l,\infty})} \right).$$

Note that by Lemmas 2.5 and 2.7, the linearisation of $\mathcal{C}^S_{k,l}$ at 0 is given by

$$\begin{aligned} & \left(d\mathcal{C}^S_{k,l} \right)_0 (\phi(s(t)), \phi_\infty, \vartheta, R) \\ = & \left(\left(d\tau^\vartheta_{k,l} \right)_0 (\phi(s(t)), \phi_\infty, \theta), \left(d\kappa^\vartheta_{k,l} \right)_0 (\phi_\infty, R) \right), \end{aligned}$$

where

$$\begin{aligned} & (6.21) \quad \left(d\tau^\vartheta_{k,l} \right)_0 (\phi(s(t)), \phi_\infty, \vartheta) \\ = \; & \mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} (\phi(s(t)) + \phi_\infty) - \frac{1}{2} g_{\omega_{k,l}(s(t))} \left(\nabla_{g_{\omega_{k,l}(s(t))}} Scal\left(\omega_{k,l}(t)\right), \nabla_{g_{\omega_{k,l}(s(t))}} \left(\phi\left(s\left(t\right)\right) + \phi_\infty\right) \right) \\ & + \frac{\partial}{\partial s}\phi(s(t)) + rk^{-1}\mathcal{L}_{X^S_\vartheta(s)}\left(\varphi_{k,l}(s)\right) + rk^{-1}\left(\mathcal{L}_{V^S_s}\left(\phi(s(t)) + \phi_\infty\right) \right) \\ & - \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) + \frac{1}{2} g_{\omega_{k,l,\infty}} \left(\nabla_{g_{\omega_{k,l,\infty}}} Scal\left(\omega_{k,l,\infty}\right), \nabla_{g_{\omega_{k,l,\infty}}} (\phi_\infty) \right) \\ & - \frac{1}{2} g_{\omega_{k,l,\infty}} \left(2rk^{-1} \nabla_{g_{\omega_{k,l,\infty}}} H_{J_\infty V_\infty}(\omega_{k,l,\infty}), \nabla_{g_{\omega_{k,l,\infty}}} (\phi_\infty) \right) - \mathcal{L}_{J_\infty X_\vartheta}\left(\varphi_{k,l,\infty}\right) \\ = \; & \frac{\partial}{\partial s}\phi(s(t)) + \mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} (\phi(s(t))) + (\mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} - \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}}) (\phi_\infty) \\ & + \frac{1}{2} g_{\omega_{k,l,\infty}} \left(\nabla_{g_{\omega_{k,l,\infty}}} \left(Scal\left(\omega_{k,l,\infty}\right) + H\left(\omega_{k,l,\infty}\right)\right), \nabla_{g_{\omega_{k,l,\infty}}} (\phi_\infty) \right) \\ & + f_S(s) \frac{1}{2} g_{\omega_{k,l}(s(t))} \left(\nabla_{g_{\omega_{k,l}(s(t))}} \left(Scal\left(\omega_{k,l}(t)\right) + H\left(\omega_{k,l}(s(t))\right)\right), \nabla_{g_{\omega_{k,l}(s(t))}} \left(\phi\left(s\left(t\right)\right) + \phi_\infty\right) \right) \\ & + (1 - f_S(s)) \frac{1}{2} g_{\omega_{k,l}(s(t))} \left(\nabla_{g_{\omega_{k,l}(s(t))}} \left(Scal\left(\omega_{k,l}(t)\right)\right), \nabla_{g_{\omega_{k,l}(s(t))}} \left(\phi\left(s\left(t\right)\right) + \phi_\infty\right) \right) \\ & + (1 - f_S(s))\, k^{-1} \mathcal{L}_{V_s} \left(\phi(s(t)) + \phi_\infty\right) + rk^{-1} \left(\mathcal{L}_{X^S_\vartheta(s)} \left(\varphi_{k,l}(s)\right) - \mathcal{L}_{J_\infty X_\vartheta} \left(\varphi_{k,l,\infty}\right) \right) \\ = \; & \frac{\partial}{\partial s}\phi(s(t)) + \mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} (\phi(s(t))) + (\mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} - \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}}) (\phi_\infty) \\ & + (1 - f_S(s))\, \mathcal{L}_{k^{-1}V_s + \nabla_{g_{\omega_{k,l}(s(t))}} \left(Scal\left(\omega_{k,l}(t)\right)\right)} \left(\phi(s(t)) + \phi_\infty\right) + rk^{-1} \left(\mathcal{L}_{X^S_\vartheta(s)} \left(\varphi_{k,l}(s)\right) - \mathcal{L}_{J_\infty X_\vartheta} \left(\varphi_{k,l,\infty}\right) \right) \\ & + \mathcal{O}(k^{-(l+1)}), \end{aligned}$$

and

$$\begin{aligned} (6.22) \qquad & \left(d\kappa^\vartheta_{k,l} \right)_0 (\phi_\infty, R) \\ = \; & \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) - \frac{1}{2} g_{\omega_{k,l,\infty}} \left(\nabla_{g_{\omega_{k,l,\infty}}} Scal\left(\omega_{k,l,\infty}\right), \nabla_{g_{\omega_{k,l,\infty}}} (\phi_\infty) \right) \\ & + \frac{1}{2} g_{\omega_{k,l,\infty}} \left(2rk^{-1} \nabla_{g_{\omega_{k,l,\infty}}} H_{J_\infty V_\infty}(\omega_{k,l,\infty}), \nabla_{g_{\omega_{k,l,\infty}}} (\phi_\infty) \right) - 2rk^{-1} H_{X_\vartheta}(\omega_{k,l,\infty}) - R \\ = \; & \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) - H_{X_\vartheta}(\omega_{k,l,\infty}) - R + \mathcal{O}(k^{-(l+1)}). \end{aligned}$$

This implies that if we define the difference operator

$$\mathcal{D}_{k,l} : W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,1,\infty}) \times L^2_{4(p+1)}(g_{k,1,\infty}) \times \mathbb{R}^{m+1} \to W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty}) \times L^2_{4p}(g_{k,1,\infty})$$

by

$$\mathcal{D}_{k,l}\left(\phi(s(t)), \phi_\infty, \vartheta, R\right) = \left(\mathcal{D}^{(1)}_{k,l}\left(\phi(s(t)), \phi_\infty, \vartheta\right), \mathcal{D}^{(2)}_{k,l}\left(\phi_\infty, \vartheta, R\right) \right), \tag{6.23}$$

where

$$\begin{aligned}
\mathcal{D}^{(1)}_{k,l}\left(\phi(s(t)),\phi_\infty,\vartheta\right) &= \frac{\partial}{\partial s}\phi(s(t)) + \mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))}\left(\phi(s(t))\right) \\
&\quad +(\mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))} - \mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}})\left(\phi_\infty\right) \\
&\quad +rk^{-1}\left(\mathcal{L}_{X^S_\vartheta(s)}\left(\varphi_{k,l}(s)\right) - \mathcal{L}_{J_\infty X_\vartheta}\left(\varphi_{k,l,\infty}\right)\right) \\
&\quad +\left(1-f_S(s)\right)\mathcal{L}_{k^{-1}V_s-\nabla_{g_{\omega_{k,l}(s(t))}}\left(Scal\left(\omega_{k,l}(t)\right)\right)}\left(\phi(s(t))+\phi_\infty\right) \\
\mathcal{D}^{(2)}_{k,l}\left(\phi_\infty,\vartheta,R\right) &= \mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}\left(\phi_\infty\right) - 2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty}) - R,
\end{aligned}$$

then by 6.21 and 6.22, with respect to the operator norm $||-||$ induced by the norm on $W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})$ we have

$$\left\|\left(d\mathcal{C}^S_{k,l}\right)_0 - \mathcal{D}_{k,l}\right\| \leq Ck^{-(l+1)}. \tag{6.24}$$

To prove Proposition 6.2 we wish to apply the following basic functional analysis lemma to $\left(d\mathcal{C}^S_{k,l}\right)_0$ and $D_{k,l}$.

Lemma 6.3. *Let $D : V \to W$ be a bounded epimorphism with bounded right inverse $\mathcal{Q}$. If $\mathcal{L}$:$V \to W$ is another linear map with*

$$\|\mathcal{L}-\mathcal{D}\| \leq \left(2\left\|\mathcal{Q}\right\|\right)^{-1},$$

then L is also an epimorphism with bounded right inverse $\mathcal{P}$ satisfying

$$\|\mathcal{P}\| \leq 2\left\|\mathcal{Q}\right\|.$$

To use this lemma, we need to know that the hypotheses apply to $\mathcal{D}_{k,l}$ and $\left\|\left(d\mathcal{C}^S_{k,l}\right)_0 - \mathcal{D}_{k,l}\right\|$. This, as well as the fact that the conclusion of this lemma suffices to give the conclusion of Proposition6.2 is a result of the following lemma combined with equation6.24.

Lemma 6.4. *The operator*

$$\mathcal{D}_{k,l} : W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4(p+1)}(g_{k,l,\infty}) \times \mathbb{R}^m \times \mathbb{R} \to W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4p}(g_{k,l,\infty})$$

is well defined, and an epimorphism of Banach spaces. There is a constant C such that for all sufficiently large k, the right inverse $\mathcal{Q}_{k,l}$ satisfies

$$\begin{aligned}
&\left\|\mathcal{Q}_{k,l}\left(\left(\psi(s(t)),\psi_\infty\right)\right)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})} \\
\leq\; & Ck^3\left\|\left(\psi(s(t)),\psi_\infty\right)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,1,\infty})\times L^2_{4p}(g_{k,l,\infty})}.
\end{aligned} \tag{6.25}$$

Proof of Lemma6.4. We may solve the two equations

$$\begin{aligned}
\mathcal{D}^{(1)}_{k,l}\left(\phi(s(t)),\phi_\infty,\vartheta\right) &= \psi(t) \\
\mathcal{D}^{(2)}_{k,l}\left(\phi_\infty,\vartheta,C\right) &= \psi_\infty,
\end{aligned}$$

for any $(\psi(t),\psi_\infty) \in W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4p}(g_{k,1,\infty})$, since we may write these equations as

$$\begin{aligned}
&\frac{\partial}{\partial s}\phi(s(t)) + \mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))}\left(\phi(s(t))\right) \\
&+\left(1-f_S(s)\right)\mathcal{L}_{k^{-1}V_s+\nabla_{g_{\omega_{k,l}(s(t))}}\left(Scal\left(\omega_{k,l}(t)\right)\right)}\left(\phi(s(t))+\phi_\infty\right) \\
=\; & -(\mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))} - \mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}})\left(\phi_\infty\right) \\
&-rk^{-1}\left(\mathcal{L}_{X^S_\vartheta(s)}\left(\varphi_{k,l}(s)\right) - \mathcal{L}_{J_\infty X_\vartheta}\left(\varphi_{k,l,\infty}\right)\right) + \psi(s(t))
\end{aligned} \tag{6.26}$$

$$\phi(0) = \phi_0$$

and

$$\begin{aligned} & \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) \\ = \ & 2rk^{-1} H_{X_\vartheta}(\omega_{k,l,\infty}) + R + \psi_\infty. \end{aligned} \tag{6.27}$$

To solve the second equation we may choose (ϑ, R) so that

$$proj_{\ker \mathfrak{D}^*_{(\omega_{k,l}(J_\infty,h))} \mathfrak{D}_{(\omega_{k,l}(J_\infty,h))}} (\psi_\infty) = -H_{X_\vartheta}(\omega_{k,l,\infty}) - R,$$

and so for this choice, the right hand side of the second equation above is orthogonal to

$$\ker \mathfrak{D}^*_{(\omega_{k,l}(J_\infty,h))} \mathfrak{D}_{(\omega_{k,l}(J_\infty,h))}.$$

We write ζ_s for the flow of the vector field

$$(1 - f_S(s)) \left(k^{-1} V_s + \nabla_{g_{\omega_{k,l}(s(t))}} (Scal(\omega_{k,l}(t))) \right).$$

Since this vector field is 0 for $s \in [2S, \infty)$, ζ_s is constant in s on this interval, and therefore we may write the pullback of this equation by ζ_s:

$$\begin{aligned} & \frac{\partial}{\partial s} \zeta_s^*(\phi(s(t))) + \mathfrak{D}^*_{\zeta_s^* \omega_{k,l}(s(t))} \mathfrak{D}_{\zeta_s^* \omega_{k,l}(s(t))} (\zeta_s^*(\phi(s(t)))) \\ = \ & -\zeta_s^* \left(\left((\mathfrak{D}^*_{\omega_{k,l}(s(t))} \mathfrak{D}_{\omega_{k,l}(s(t))} - \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}}) (\phi_\infty) \right)^\perp + rk^{-1} \left(\mathcal{L}_{X^S_\vartheta(s)} (\varphi_{k,l}(s)) - \mathcal{L}_{J_\infty X_\vartheta} (\varphi_{k,l,\infty}) \right)^\perp \right) \\ & \zeta_s^* \left(- \left((1 - f_S(s)) \mathcal{L}_{k^{-1} V_s - \nabla_{g_{\omega_{k,l}(s(t))}} (Scal(\omega_{k,l}(t)))} (\phi_\infty) \right)^\perp + \psi(t) \right), \end{aligned}$$

so writing $\widetilde{\phi}(s(t)) = \phi \circ \zeta_s$, and since ζ_s is constant for $s \in [2S, \infty)$, and in particular bounded, we obtain an equation of the form

$$\frac{\partial}{\partial s} \widetilde{\phi}(s(t)) + \mathfrak{D}^*_{\zeta_s^* \omega_{k,l}(s(t))} \mathfrak{D}_{\zeta_s^* \omega_{k,l}(s(t))} \left(\widetilde{\phi}(s(t)) \right) = \widetilde{\psi}(t)$$

where $\widetilde{\psi}(t) \in W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})$ and where $\widetilde{\psi}(t)$ is L^2 orthogonal to the kernel of $\mathfrak{D}^*_{\zeta^*_\infty \omega_{k,l,\infty}} \mathfrak{D}_{\zeta^*_\infty \omega_{k,l,\infty}}$, and therefore this equation has a solution by Theorem 7.10. and therefore, Equation 6.26 has a solution $\phi(s)$ in the space $W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})$.

This shows that $\mathcal{D}_{k,l}$ is surjective. We therefore obtain a right inverse to $\mathcal{Q}_{k,l}$ to $\mathcal{D}_{k,l}$ defined by

$$\mathcal{Q}_{k,l} (\psi(t), \psi_\infty) = (\phi(t), \phi_\infty, \vartheta, R),$$

where

$$-2rk^{-1} H_{X_\vartheta}(\omega_{k,l,\infty}) - R = \psi_\infty^\perp$$

and where the pair $(\phi(t), \phi_\infty)$ solves equations 6.26 and 6.27 respectively. It remains to prove the estimate 6.25. Note that by Lemma 41 of [B], there is a constant C, such that for our choice of (ϑ, R),

$$\begin{aligned} \|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})} & \leq C \left(\|\phi_\infty\|_{L^2(g_{k,l,\infty})} + \left\| \mathcal{D}^{(2)}_{k,l} (\phi_\infty, \vartheta, R) \right\|_{L^2_{4p}(g_{k,l,\infty})} \right) \\ & = C \left(\|\phi_\infty\|_{L^2(g_{k,l,\infty})} + \left\| \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) - 2rk^{-1} H_{X_\vartheta}(\omega_{k,l,\infty}) - R \right\|_{L^2_{4p}(g_{k,l,\infty})} \right), \end{aligned}$$

and by Lemma 39 of [B], there is also an estimate

$$\|\phi_\infty\|_{L^2(g_{k,l,\infty})} \leq Ck^3 \left\| \mathfrak{D}^*_{\omega_{k,l,\infty}} \mathfrak{D}_{\omega_{k,l,\infty}} (\phi_\infty) - 2rk^{-1} H_{X_\vartheta}(\omega_{k,l,\infty}) - R \right\|_{L^2_{4p}(g_{k,l,\infty})} \tag{6.28}$$

and therefore obtain an estimate

$$\begin{aligned}\|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})} &\leq Ck^3\left\|\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}(\phi_\infty)-2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty})-R\right\|_{L^2(g_{k,l,\infty})}\\ &\quad +C\left\|\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}(\phi_\infty)-2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty})-R\right\|_{L^2_{4p}(g_{k,l,\infty})}\\ &\leq Ck^3\left\|\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}(\phi_\infty)-2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty})-R\right\|_{L^2_{4p}(g_{k,l,\infty})}\\ &= Ck^3\|\psi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}\end{aligned}$$

By Lemma 4.20 above, for the choice of solution to the initial value problem 6.26 where the initial condition is set to $\phi_0=0$, we also have an estimate

$$\left\|(\widetilde{\phi}(s(t))\right\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})}\leq C\left(\left\|\widetilde{\psi}(s(t))\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))}\right),$$

and since the ζ_s is bounded, we obtain an estimate:

$$\begin{aligned}&\|(\phi(s(t))\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})}\\ \leq\ & C\left(\left\|\left(\left(\mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))}-\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}\right)(\phi_\infty)\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}+\|\psi(s(t))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}))}\right)\\ &+C\left\|rk^{-1}\left(\mathcal{L}_{X^S_\vartheta(s)}(\varphi_{k,l,\infty})-\mathcal{L}_{J_\infty X_\vartheta}(\varphi_{k,l,\infty})\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))}\\ &+C\left\|\left((1-f_S(s))\,\mathcal{L}_{k^{-1}V_s-\nabla g_{\omega_{k,l}(s(t))}}\left(Scal\big(\omega_{k,l}(t)\big)\right)(\phi_\infty)\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))}.\end{aligned}$$

Clearly we have estimates

$$\begin{aligned}&\left\|\left((1-f_S(s))\,\mathcal{L}_{k^{-1}V_s-\nabla g_{\omega_{k,l}(s(t))}}\left(Scal\big(\omega_{k,l}(t)\big)\right)(\phi_\infty)\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))}\\ \leq\ & C\|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})}\leq Ck^3\|\psi_\infty\|_{L^2_{4p}(g_{k,l,\infty})},\\ &\left\|\left(\left(\mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))}-\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}\right)(\phi_\infty)\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}\\ \leq\ & C\|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})}\leq Ck^3\|\psi_\infty\|_{L^2_{4p}(g_{k,l,\infty})},\end{aligned}$$

since $(1-f_S(s))$ is supported on a finite interval, and the operator norm

$$\left\|\mathfrak{D}^*_{\omega_{k,l}(s(t))}\mathfrak{D}_{\omega_{k,l}(s(t))}-\mathfrak{D}^*_{\omega_{k,l,\infty}}\mathfrak{D}_{\omega_{k,l,\infty}}\right\|$$

has finite integral when multiplied by the weight function, and where we have also used estimate 6.28. We may write

$$\begin{aligned}&rk^{-1}\left(\mathcal{L}_{X^S_\vartheta(s)}(\varphi_{k,l,\infty})-\mathcal{L}_{J_\infty X_\vartheta}(\varphi_{k,l,\infty})\right)\\ =\ & rk^{-1}(f_S(s)-1)\,\mathcal{L}_{X^S_\vartheta(s)}(\varphi_{k,l,\infty})\\ =\ & (f_S(s)-1)\,2rk^{-1}H_{X_\vartheta}(\omega_{k,l,\infty})\\ =\ & (1-f_S(s))\left(\psi^\perp_\infty+R\right),\end{aligned}$$

so that

$$\left\|rk^{-1}\left(\mathcal{L}_{X^S_\vartheta(s)}(\varphi_{k,l,\infty})-\mathcal{L}_{J_\infty X_\vartheta}(\varphi_{k,l,\infty})\right)^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))}$$

$$= \left\|(1-f_S(s))\,\psi_\infty^\perp\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty}))} \leq C\left\|\psi_\infty\right\|_{L^2_{4p}(g_{k,l,\infty})},$$

since again $(1-f_S(s))$ is supported on a finite interval. These estimates then combine to give:

$$\|(\phi(s(t))\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})} \leq C\left(\|\psi(s(t))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}))} + k^3\left\|\psi_\infty\right\|_{L^2_{4p}(g_{k,l,\infty})}\right)$$

Note also that we have by construction (since $H_{X_\vartheta}(\omega_{k,l,\infty})$ is L^2 orthogonal to the constants) that

$$\begin{aligned}
\left\|\psi_\infty^\perp\right\|_{L^2_{4p}(g_{k,l,\infty})} &\geq C\left(\left\|H_{X_\vartheta}(\omega_{k,l,\infty})\right\|_{L^2_{4p}(g_{k,l,\infty})} + |R|\right)\\
&\geq C\left(\|\vartheta\|_{L^2_{4p}(g_{k,l,\infty})} + |R|\right)\\
&\geq C\left(\|\vartheta\| + |R|\right),
\end{aligned}$$

by formula 6.13 and the argument of Lemma 4.15.

Then finally we obtain the estimate

$$\begin{aligned}
&\left\|\mathcal{Q}_{k,l}\left(\psi(t),\psi_\infty\right)\right\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}}\\
=\;&\left\|(\phi(t),\phi_\infty,\vartheta,R)\right\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}}\\
=\;&\|\phi(t)\|_{W_{4(p+1),q,w_\varepsilon(s)}(g_{k,l,\infty})} + \|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})} + ||\vartheta|| + |R|\\
\leq\;& C\left(\|\psi(s(t))\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}))} + k^3\left\|\psi_\infty\right\|_{L^2_{4p}(g_{k,l,\infty})}\right)\\
\leq\;& Ck^3\left(\left\|(\psi(s(t),\psi_\infty)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}))\times L^2_{4p}(g_{k,l,\infty})}\right),
\end{aligned}$$

as stated. □

Finally we have the

Proof of Proposition 6.2. By Lemma 6.4 we have an estimate on the operator norm

$$\|\mathcal{Q}_{k,l}\| \leq Ck^3,$$

so in order to apply Lemma6.3 to $(d\mathcal{C}^S_{k,l})_0 - \mathcal{D}_{k,l}$, we need that

$$\left\|(d\mathcal{C}^S_{k,l})_0 - \mathcal{D}_{k,l}\right\| \leq Ck^{-3}.$$

By estimate 6.24, this will be acheived whenever $l \geq 3$. The result follows immediately. □

6.3. **An estimate on the non-linear term.**

As in the sketch in Section 6.1 we define $\mathcal{N}^S_{k,l} := \mathcal{C}^S_{k,l} - d\mathcal{C}^S_{k,l}$. This is the analogue of Lemma 7.1 in [F] Lemma 44 in [B].

Proposition 6.5. *Let $k \geq 3$. There are positive constants c and K, such that for all*

$$\left(\phi(s),\phi_\infty,\vartheta_1,R_1\right),\left(\psi(s),\psi_\infty,\vartheta_2,R_2\right) \in W^0_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4(p+1)}(g_{k,l,\infty}) \times \mathbb{R}^m \times \mathbb{R}$$

with

$$\begin{aligned}
\left\|(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}} &\leq c,\\
\left\|(\psi(s),\psi_\infty,\vartheta_2,R_2)\right\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}} &\leq c,
\end{aligned}$$

and for k sufficiently large,

$$\begin{aligned}
&\left\|\mathcal{N}^S_{k,l}\left(\varrho(s),\varrho_\infty,\vartheta_1,R_1\right) - \mathcal{N}^S_{k,l}\left(\psi(s),\psi_\infty,\vartheta_2,R_2\right)\right\|\\
\leq\;& K\max\left\{\left\|(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\right\|,\left\|\psi(s),\psi_\infty,\vartheta_2,R_2\right\|\right\}\left\|(\varrho(s)-\psi(s),\varrho_\infty-\psi_\infty,\vartheta_1-\vartheta_2,R_1-R_2)\right\|
\end{aligned}$$

where on the left hand side the norm is the norm on $W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}$, *and on the right hand side, the norms are the norm on* $W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}$.

Proof. If we write

$$\begin{aligned}
&[(\phi(s),\phi_\infty,\vartheta_1,R_1)\,,(\psi(s),\psi_\infty,\vartheta_2,R_2)]\\
=\;&\{w\,(\phi(s),\phi_\infty,\vartheta_1,R_1)+(1-w)(\psi(s),\psi_\infty,\vartheta_2,R_2)|\; w\in[0,1]\}\,,
\end{aligned}$$

then by the mean value theorem,we have

$$\begin{aligned}
&\left\|\mathcal{N}^S_{k,l}\,(\phi(s),\phi_\infty,\vartheta_1,R_1)-\mathcal{N}^S_{k,l}\,(\psi(s),\psi_\infty,\vartheta_2,R_2)\right\|\\
\leq\;&\sup_{(\chi(s),\chi_\infty,\vartheta,R)}\left\|\left(d\mathcal{N}^S_{k,l}\right)_{(\chi(s),\chi_\infty,\vartheta,R)}\right\|\|(\phi(s)-\psi(s),\phi_\infty-\psi_\infty,\vartheta_1-\vartheta_2,R_1-R_2)\|
\end{aligned}$$

where the sup is over all $(\chi(s),\chi_\infty,\vartheta,R)\in[(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\,,(\psi(s),\psi_\infty,\vartheta_2,R_2)]$.

By construction, we have

$$\begin{aligned}
\left(d\mathcal{N}^S_{k,l}\right)_{(\chi(s),\chi_\infty,\vartheta,R)}&=d\left(\mathcal{C}^S_{k,l}\right)_{(\chi(s),\chi_\infty,\vartheta,R)}-\left(d\mathcal{C}^S_{k,l}\right)_0\\
&=(d\,(\tau_{k,l})_{(\chi(s),\chi_\infty,\vartheta)}-d(\tau_{k,l})_0,d(\kappa_{k,l})_{(\chi_\infty,\vartheta,R)}-d(\kappa_{k,l})_{(\chi_\infty,\vartheta,R)}).
\end{aligned}$$

Using formulas 6.15 and 6.16 we may calculate the directional derivatives of $\tau_{k,l}$ and $\kappa_{k,l}$ at $(\chi(s),\chi_\infty,\vartheta,R)$ and 0 respectively, in the direction of $\left(\phi(s),\phi_\infty,\vartheta',R'\right)$ to obtain

$$\begin{aligned}
&\left(d\,(\tau_{k,l})_{(\chi(s),\chi_\infty,\vartheta)}-d(\tau_{k,l})_0\right)\left(\phi(s),\phi_\infty,\vartheta'\right)\\
=\;&\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,l}(t)}-Scal_{\omega_{k,l,,\infty}}\right)(\phi(s),\phi_\infty)\\
&+\mathcal{L}_{X^S_\vartheta(s)}\,(\phi(s)+\phi_\infty)-\mathcal{L}_{J_\infty X_\vartheta}\,(\phi_\infty)+\mathcal{L}_{X^S_{\vartheta'}(s)}\,(\chi(s)+\chi_\infty)-\mathcal{L}_{J_\infty X_{\vartheta'}}\,(\chi_\infty)\\
=\;&\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,l}(t)}-Scal_{\omega_{k,l,,\infty}}\right)(\phi(s),\phi_\infty)\\
&+\mathcal{L}_{X^S_\vartheta(s)}\,(\phi(s))+\left(\mathcal{L}_{X^S_\vartheta(s)}-\mathcal{L}_{J_\infty X_\vartheta}\right)(\phi_\infty)+\mathcal{L}_{X^S_{\vartheta'}(s)}\,(\chi(s)+\chi_\infty)-\mathcal{L}_{J_\infty X_{\vartheta'}}\,(\chi_\infty)\,,
\end{aligned}$$

$$\begin{aligned}
&\left(d(\kappa_{k,l})_{(\chi_\infty,\vartheta,R)}-d(\kappa_{k,l})_0\right)\left(\phi_\infty,\vartheta',R'\right)\\
=\;&\left(d_{(\chi_\infty,\vartheta,R)}-d_0\right)Scal_{\omega_{k,l,,\infty}}\,(\phi_\infty)\\
&+\mathcal{L}_{J_\infty X_\vartheta}\,(\phi_\infty)+\mathcal{L}_{J_\infty X_{\vartheta'}}\,(\chi_\infty)\,,
\end{aligned}$$

so that we obtain

$$\begin{aligned}
&\left\|\left(d\mathcal{N}^S_{k,l}\right)_{(\chi(s),\chi_\infty,\vartheta,R)}\left(\phi(s),\phi_\infty,\vartheta',R'\right)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}}\\
\leq\;&\left\|\left(d_{(\chi(t),\chi_\infty)}-d_0\right)\left(Scal_{\omega_{k,l}(t)}-Scal_{\omega_{k,l,\infty}}\right)(\phi(s),\phi_\infty)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}\\
&+\left\|\mathcal{L}_{X^S_\vartheta(s)}\,(\phi(s))\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}+\left\|\left(\mathcal{L}_{X^S_\vartheta(s)}-\mathcal{L}_{J_\infty X_\vartheta}\right)(\phi_\infty)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}\\
&+\left\|\mathcal{L}_{X^S_{\vartheta'}(s)}\,(\chi(s)+\chi_\infty)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}+\left\|\mathcal{L}_{J_\infty X_{\vartheta'}}\,(\chi_\infty)\right\|_{W_{4,p,q-1,w_\varepsilon(s)}(g_{k,l,\infty})}\\
&+\left\|(d_{\chi_\infty}-d_0)\,Scal_{\omega_{k,l,,\infty}}\,(\phi_\infty)\right\|_{L^2_{4p}(g_{k,l,\infty})}+\|\mathcal{L}_{J_\infty X_\vartheta}\,(\phi_\infty)\|_{L^2_{4p}(g_{k,l,\infty})}+\left\|\mathcal{L}_{J_\infty X_{\vartheta'}}\,(\chi_\infty)\right\|_{L^2_{4p}(g_{k,l,\infty})}\\
\leq\;&K\left(\|(\chi(t),\chi_\infty)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})}\cdot\|(\phi(s),\phi_\infty)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4(p+1)}(g_{k,l,\infty})}\right)
\end{aligned}$$

$$
\begin{aligned}
&+K\left(\|\vartheta\|\cdot\|\phi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}+\left\|\vartheta'\right\|\cdot\left(\|\chi(s)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})}+\|\chi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}\right)\right)\\
&+K\left(\left\|\vartheta'\right\|\cdot\|\chi(s)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})}+\|\chi_\infty\|\cdot\|\phi_\infty\|_{L^2_{4(p+1)}(g_{k,l,\infty})}\right)\\
&+K\left(\|\vartheta\|\cdot\|\phi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}+\left\|\vartheta'\right\|\cdot\|\chi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}\right)\\
\leq\ & K\left(\|(\chi(s))\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})}+\|\chi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}+\|\vartheta\|\right)\\
&\times\left(\|(\phi(s))\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})}+\|\phi_\infty\|_{L^2_{4p}(g_{k,l,\infty})}+\left\|\vartheta'\right\|\right)\\
\leq\ & K\left(\|(\chi(s),\chi_\infty,\vartheta,R)\|\cdot\left\|\left(\phi(s),\phi_\infty,\vartheta',R'\right)\right\|\right)\\
\leq\ & \max\{\|(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\|,\|(\psi(s),\psi_\infty,\vartheta_2,R_2)\|\}\cdot\left\|\left(\phi(s),\phi_\infty,\vartheta',R'\right)\right\|,
\end{aligned}
$$

where we have used the bound

$$
\begin{aligned}
&\|(\chi(s),\chi_\infty,\vartheta,R)\|_{W_{4,p+1,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})\times\mathbb{R}^m\times\mathbb{R}}\\
\leq\ & \max\{\|(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\|,\|(\psi(s),\psi_\infty,\vartheta_2,R_2)\|\}\leq c,
\end{aligned}
$$

and also Lemma 4.19, where we note that the proof of the latter works just as well for the metrics $g_{k,l,\infty}$ as for $g_{k,1,\infty}$. We therefore obtain a uniform bound bound on the operator norm

$$
\left\|\left(d\mathcal{N}^S_{k,l}\right)_{(\chi(s),\chi_\infty,\vartheta,R)}\right\|\leq\max\{\|(\varrho(s),\varrho_\infty,\vartheta_1,R_1)\|,\|(\psi(s),\psi_\infty,\vartheta_2,R_2)\|\},
$$

and the result follows. □

6.4. **Proof of Theorem 1.3.** Clearly, Propositions 6.2 and 6.5 establish the following two properties.

(i) The derivative of the map $\mathcal{C}^S_{k,l}$ at 0 is an epimorphism, whose right inverse $\mathcal{P}_{k,l}$ which enjoys a uniform estimate

$$
\|\mathcal{P}_{k,l}\left((\psi_t,\psi_\infty)\right)\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,1,\infty})\times\mathbb{R}^m\times\mathbb{R}}\leq Ck^3\|(\psi_t,\psi_\infty)\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty})\times L^2_{4p}(g_{k,l,\infty})}
$$

(ii) The non-linear part of $\mathcal{C}^S_{k,l}$, namely $\mathcal{N}_{k,l}:=\mathcal{C}^S_{k,l}-d\mathcal{C}^S_{k,l}$ has the property that there exists a constant C such that for all sufficiently small M, $\mathcal{N}_{k,l}$ is Lipschitz with constant M on a ball of radius CM.

Given these facts, the existence of a solution to equation 6.18 follows. Namely, points (i) and (ii) above combine to say that the radius δ_k' of the ball $B_{\delta_k'}(0)$ on which $\mathcal{N}_{k,l}$ is Lipschitz with constant $\frac{2}{\|\mathcal{P}_{k,l}\|}$, is bounded below by

$$
\frac{2C}{\|\mathcal{P}_{k,l}\|}\geq Ck^{-3}. \tag{6.29}
$$

Then defining

$$
\delta_k=\delta_k'\left(\frac{2}{\|\mathcal{P}_{k,l}\|}\right) \tag{6.30}
$$

as in Theorem 6.1, we obtain that

$$
\delta_k\geq\frac{4C}{\|\mathcal{P}_{k,l}\|^2}\geq Ck^{-6}, \tag{6.31}
$$

and therefore by Theorem 6.1,

$$\left\|\mathcal{C}^S_{k,l}(0) - (\psi_t, \psi_\infty)\right\|_{W_{4,p,q,w_\varepsilon(s)}(g_{k,l,\infty}) \times L^2_{4p}(g_{k,l,\infty})} \leq Ck^{-6}$$

implies that there is a solution $(\phi(s(t)), \phi_\infty, \vartheta, R)$ to

$$\mathcal{C}^S_{k,l}(\phi(s(t)), \phi_\infty, \vartheta, R) = (\psi(s(t)), \psi_\infty). \tag{6.32}$$

In particular, by the Sobolev bound 6.17, for $l \geq 6$, Equation 6.18, has a solution for every S, and therefore since up to time S a solution to this equation is equivalent a solution to 6.3, the latter will have a solution for all time.

7. Appendix

7.1. **Linear parabolic equations on compact Riemannian manifolds.** In this appendix we will state and sketch the proofs of the existence, uniqueness and regularity theorems, for linear parabolic PDEs on compact Riemannian manifolds. These theorems are probably more or less standard, but it seems difficult to find precise statements and proofs of them in the literature. One source is Huisken and Polden, and we will follow their basic approach here, but our treatment will be slightly more streamlined, and we will also modify the norms that are involved to accomodate our particular problem.

7.1.1. *Notation and basic definitions.* Throughout this appendix we will let

$$(E, \langle -, - \rangle) \to (M, g)$$

be a smooth complex vector bundle over a Riemannian manifold with an Hermitian metric $\langle -, - \rangle$ on E. In practice, E will be either the endomorphism bundle of another vector bundle, or the trivial line bundle. We will consider the theory of equations of the form

$$\frac{\partial u(t)}{\partial t} + L_t u(t) = f(t),$$

where $L(t) : \Gamma(E) \to \Gamma(E)$ is a 1-parameter family of differential operators of order $2d$. We will assume that that L_t is self-adjoint and strongly elliptic for each t. Recall that self-adjoint means that $\langle L_t u, v\rangle = \langle u, L_t v\rangle$ for all $u, v \in E_x$, and all $x \in M$. To define strongly elliptic, we recall the definition of the symbol of a differential operator. If $L : C^\infty(E) \to C^\infty(E)$ is a differential operator, then for each $u \in \Gamma(E)$ in local coordinates we have

$$Lu = \sum_{|\alpha| \leq 2d} L^\alpha \frac{\partial^\alpha u}{\partial x^\alpha}$$

where $L^\alpha : E \to E$ is a bundle endomorphism and $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index. The **principal symbol** is given by $\sigma(L) : E \otimes T^*M \to E$ is defined by

$$\sigma(L, x)(\xi)(v) = \sum_{|\alpha| = 2d} \xi_{i_1}^{\alpha_1} \cdots \xi_{i_{2d}}^{\alpha_{2d}} L^\alpha(v)$$

where $x \in M$ is any point and $0 \neq \xi \in \Gamma(T^*_x M)$, written locally as $\xi = \xi_i dx^i$. Then L is called **strongly elliptic** (or sometimes uniformly strongly elliptic) if there is a constant c such that for all $x \in M$ and all $0 \neq \xi \in \Gamma(T^*_x M)$

$$Re\left(\langle \sigma(L, x)(\xi)(v), v\rangle\right) \geq c\, |\xi|^{2d}$$

for all $0 \neq v \in E_x$. Note that this implies in particular that for each $x \in M$ and $0 \neq \xi \in \Gamma(T^*_x M)$, $\sigma(L, x)(\xi) : E_x \to E_x$ is an isomorphism, which is usually taken as the definition of an elliptic operator.

Remark 7.1. From here on out, we will also assume that the set $\ker(L_t) = K \subset C^\infty(E)$ is independent of t. This is of course trivially satisfied when $L_t = L$ is just a constant path of operators.

7.1.2. *Some results from functional analysis.* We will also have need of **Gårding's Inequality**, which states that for a a strongly elliptic operator L of order $2d$ there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that for every $u \in L^2_d(E)$,

$$Re \langle Lu, u\rangle_{L^2} \geq C_1 \|u\|^2_{L^2_d} - C_2 \|u\|^2_{L^2}.$$

If there furthermore exists an $M > 0$ such that

$$Re \langle Lu, u\rangle \geq M \|u\|^2_{L^2},$$

for all $u \in L^2_d(E)$ then by Gårding's inequality

$$Re \langle Lu, u\rangle_{L^2} \geq C \|u\|^2_{L^2_d},$$

where $C = \frac{C_1}{\left(1+\frac{C_2}{M}\right)}$. In this case we say that L is **positive definite**. Note that if L is positive definite, then so is L^*L, and therefore

$$\|Lu\|^2_{L^2} \geq C \|u\|^2_{L^2_{2d}}.$$

In particular, if L is positive definite, the $\ker L = 0$. If L is also self-adjoint then $cokerL = \ker L^* = \ker L = 0$, so L is invertible. We will call L **positive semi-definite** if $\mathrm{Re}\langle Lu, u\rangle_{L^2} \geq 0$. If L is positive semi-definite, then clearly $L + Id$ is positive definite.

For a complex Hilbert space H with norm $\|\cdot\|_H$, and $H' \subset H$ a linear sub-space with norm $\|\cdot\|_{H'}$, such that the inclusion $H' \hookrightarrow H$ is continuous. The key result we will need to prove the parabolic existence, uniqueness, and regularity theorems is the following result of Lax-Lions-Milgram.

Theorem 7.2. *(Lax-Lions-Milgram) Let $B : H \times H' \to \mathbb{C}$ be a sesquilinear form with the following properties*

1. ***Continuity****. For all fixed $\phi \in H'$, the map given by $v \mapsto B(v, \phi)$ is a continuous linear map $H \to \mathbb{C}$.*

2. ***Coercivity****. There is a constant $\lambda > 0$ such that for all $\phi \in H'$, $ReB(\phi, \phi) \geq \lambda \|\phi\|^2_H$.,*

Then for any continous linear map $F : (H', \|\cdot\|_{H'}) \to \mathbb{R}$, there exists $v \in H$ such that for all $\phi \in H'$, $B(v, \phi) = F(\phi)$. Furthermore $\|v\|_H \leq \frac{c}{\lambda} \|F\|$, where $\|F\|$ denotes the operator norm.

7.1.3. *Parabolic Sobolev norms.* Now we will introduce the norms that will be used for parabolic theory. The primitive form of the norm will be defined as follows. Let $w_\epsilon(t)$ be a smooth, real-valued, weight function (to be defined later), and define the **parabolic Sobolev norm** $|| - ||_{V_{2k,w_\varepsilon(t)}}$ of compactly supported function smooth function $f \in C^\infty_0(M \times [0, \infty))$ by

$$\|f\|^2_{V_{2k,w_\varepsilon(t)}} = \int_0^\infty |w_\varepsilon(t)|^2 \|f\|^2_{L^2_{2k}} dt.$$

To prove this is a norm, the only (slightly) non-trivial property to check is the triangle inequality. This follows from the triangle inequality for$\|-\|_{L^2_{2k}}$ and Hölder's inequality on $[0, \infty)$. Namely:

$$\begin{aligned}
\|f+g\|^2_{V_{2k,w_\varepsilon(t)}} &= \int_0^\infty |w_\varepsilon(t)|^2 \|f+g\|^2_{L^2_{2k}} dt \leq \int_0^\infty |w_\varepsilon(t)|^2 \left(\|f\|_{L^2_{2k}} + \|g\|_{L^2_{2k}}\right)^2 dt \\
&= \int_0^\infty \left(|w_\varepsilon(t)|^2 \|f\|^2_{L^2_{2k}} + 2 |w_\varepsilon(t)|^2 \|f\|_{L^2_{2k}} \|g\|_{L^2_{2k}} + |w_\varepsilon(t)|^2 \|g\|^2_{L^2_{2k}}\right) dt \\
&\leq \|f\|^2_{V_{2k,w_\varepsilon(t)}} + 2 \|g\|^2_{V_{2k,w_\varepsilon(t)}} \|g\|^2_{V_{2k,w_\varepsilon(t)}} + \|g\|^2_{V_{2k,w_\varepsilon(t)}}
\end{aligned}$$

$$= \quad (\|f\|_{V_{2k,w_\varepsilon(t)}} + \|g\|_{V_{2k,w_\varepsilon(t)}})^2.$$

Now we fix a subspace $K \subset E$

Definition 7.3. *We define the* ***the primitive parabolic Sobolev space with weight*** $w_\varepsilon(t)$ ***relative to*** K *to be the completion* $V_{2k,w_\varepsilon(t)}(E,K)$ *of the space* $K^\perp \cap C_0^\infty(E, M \times [0,\infty))$ *with respect to this norm, where* $K^\perp \subset L^2((E, M \times [0,\infty))$ *is the* L^2 *orthogonal complement of* K*.*

To obtain strong solutions of our parobolic equations, we will need to develop a regularity theory for solutions of parabolic equations, which will require a more sophisticated version of this norm. Namely, for each triple of non-negative integers d, p, and q with $q \leq p$ and a smooth function weight function $w_\varepsilon(t)$ on the interval $[0,\infty)$ with $w_\varepsilon(0) = 0$, we define a family of **parabolic Sobolev norms** $\|\cdot\|_{W_{d,p,q,w_\varepsilon(t)}}$ on the space $C_0^\infty(M \times [0,\infty), p_M^*(E))$ of compactly supported sections of the bundle $p_M^*(E)$ over $M \times [0,\infty)$ (where $p_M : M \times [0,\infty) \to M$ is the natural projection) by:

$$\|\phi(t)\|^2_{W_{d,p,,q,w_\varepsilon(t)}(g)} = \sum_{j=0}^{q} \int_0^\infty |w_\varepsilon(t)|^2 \left\| \frac{\partial^j \phi(t)}{\partial t} \right\|^2_{L^2_{d(p-j)}(g)} dt = \sum_{j=0}^{q} \left\| \frac{\partial^j \phi(t)}{\partial t} \right\|^2_{V_{d(p-j)}(g)}$$

For $\phi(t) \in C_0^\infty(M \times [0,\infty), p_M^*(E))$ this is clearly finite and, is a norm since $\|-\|^2_{V_{2d(p-j)}(g)}$ is a norm and $\frac{\partial^j}{\partial t}$ is linear. We will furthermore set

$$\|\cdot\|_{W_{d,p,w_\varepsilon(t)}} := \|\cdot\|_{W_{d,p,p,w_\varepsilon(t)}}.$$

Let $K \subset E$ be a fixed subspace.

Definition 7.4. We define the **parabolic Sobolev spaces with weight** $w_\varepsilon(t)$ **relative to** K to be the completion of $K^\perp \cap C_0^\infty(M \times [0,\infty), p_M^*(E))$ with respect to these norm, where $K^\perp \subset L^2(E)$ is the L^2-orthogonal complement of K. We will denote this space by $W_{d,p,s,w_\varepsilon(t)}(E,K)$.

Remark 7.5. In practice $K \subset E$ will be the kernel of the operator L_t, which we will assume to be independent of t.

Remark 7.6. In order to make the parabolic theory work in our setting we will define the weight function to be $w_\varepsilon(t) = e^{-\eta(t)} t^{-\varepsilon\psi(t)}$, where $0 < \varepsilon < \infty$ is a positive real number and $\eta(t)$ and $\psi(t)$ are smooth functions defined below, which in particular will make $w_\varepsilon(t)$ a smooth function with $w_\varepsilon(0) = 1$.

For our purposes we will need to impose the further restriction that the Sobolev norms in the definition of the $W_{d,p,w_\varepsilon(t)}$ norm vanish at infinity.

Definition 7.7. We define the space $W^0_{d,p,q,w_\varepsilon(t)}(E,K) \hookrightarrow W_{d,p,q,w_\varepsilon(t)}(E,K)$ to be the subset

$$W^0_{d,p,q,w_\varepsilon(t)}(E,K) = \{\phi_t \in W_{d,p,q,w_\varepsilon(t)}(E,K) | \lim_{t\to\infty} ||\partial_t^j(\phi_t)||_{L^2_{d(p-j)}} = 0, \text{ for all } 0 \leq j \leq q\}.$$

Lemma 7.8. *For* $p \geq 1$*, the subset* $W^0_{d,p,q,w_\varepsilon(t)}(E,K)$ *is a closed subspace of* $W_{d,p,q,w_\varepsilon(t)}(E,K)$*, and therefore a Banach space.*

Proof. $W^0_{d,p,q,w_\varepsilon(t)}(E,K)$ is clearly a subspace, so it remains only to show that it is closed.

First of all, for a path $\phi_t \in W_{2,p,w_\varepsilon(t)}(E,K)$. Thinking of $\partial_t^j(\phi_t)$ as a map $\partial_t^j(\phi) : [0,\infty) \to L^2_{d(p-j)}(X)$, since $w_\varepsilon(t)$ is smooth we have that for any finite number S, and each $0 \leq j < q$, $\partial_t^j(\phi_t) \in L^2_1([0,S], L^2_{d(p-j)}(X)) \hookrightarrow C^1([0,S], L^2_{d(p-j)}(X))$, by the (Banach space valued) Sobolev embedding theorem, and therefore ϕ_t has p strong time derivatives. In particular, for each $0 \leq j \leq q$

$$||\partial_t^j(\phi_t)||_{L^2_{d(p-j)}}$$

is continuous as a function on $[0, \infty)$.

Now take a sequence $\phi_{t,i} \in W^0_{d,p,qw_\varepsilon(t)}(E, K)$, and suppose $\phi_{t,i} \to \phi_t$ in the norm $|| - ||_{W_{d,p,q,w_\varepsilon(t)}}$ for some $\phi_t \in W_{d,p,q,w_\varepsilon(t)}(M, E)$. Explicitly this means that

$$\lim_{j \to \infty} ||\phi_{t,i} - \phi_t||_{W_{d,p,q,w_\varepsilon(t)}} = \sum_{j=0}^{q} \lim_{i \to \infty} \int_0^\infty |w_\varepsilon(t)|^2 \left\| \partial_t^j(\phi_{t,i} - \phi_t) \right\|^2_{L^2_{2d(p-j)}(g)} = 0.$$

In other words

$$|w_\varepsilon(t)|^2 \left\| \partial_t^j(\phi_{t,i} - \phi_t) \right\|^2_{L^2_{2d(p-j)}(g)} \to 0$$

in $L^1([0, \infty))$ for each j. Therefore there is a subsequence (which we still denote by $\phi_{t,i}$), which converges pointwise almost everywhere. Since $|w_\varepsilon(t)|^2 > 0$, this means that for almost every $t \in [0, \infty)$, we have

$$\lim_{i \to \infty} \left\| \partial_t^j(\phi_{t,i} - \phi_t) \right\|^2_{L^2_{2d(p-j)}(g)} = 0$$

for each j.

Now fix $\delta > 0$. Then there exists an i such that

$$\left\| \partial_t^j(\phi_{t,i} - \phi_t) \right\|_{L^2_{2d(p-j)}(g)} \leq \delta/2.$$

Since $\phi_{t,i} \in W^0_{d,p,w_\varepsilon(t)}(E, K)$ we may find $T >> 0$ such that for $t > T$, we have

$$\left\| \partial_t^j(\phi_{t,i}) \right\|_{L^2_{2d(p-j)}(g)} \leq \delta/2$$

for each j. Then for almost all $t > T$ we have

$$\left\| \partial_t^j(\phi_t) \right\|_{L^2_{2d(p-j)}(g)} \leq \left\| \partial_t^j(\phi_{t,i} - \phi_t) \right\|_{L^2_{2d(p-j)}(g)} + \left\| \partial_t^j(\phi_{t,i}) \right\|_{L^2_{2d(p-j)}(g)} \leq \delta,$$

for each j. Since $\left\| \partial_t^j(\phi_t) \right\|_{L^2_{2d(p-j)}(g)}$ is continuous, this must hold for all $t > T$. Therefore

$$\lim_{i \to \infty} \left\| \partial_t^j(\phi_t) \right\|_{L^2_{2d(p-j)}(g)} = 0$$

and $\phi_t \in W^0_{d,p,q,w_\varepsilon(t)}(M, K)$, and $W^0_{d,p,q,w_\varepsilon(t)}(M, K)$ is closed. Since a closed subspace of a Banach space is a Banach space, this implies that $W^0_{d,p,q,w_\varepsilon(t)}(E, K)$ is a Banach space. □

7.1.4. *Linear equations with a time dependent operator.* We will now prove state and prove a low-regularity version of the existence for solutions of linear parabolic equations whose forcing term lies in a parabolic Sobolev space.

Theorem 7.9. *Let L_t be a self-adjoint, strongly elliptic, semi-definite operator of order $2k$ for all t, assume that L_t is a smooth family, and assume that $\left\| \frac{\partial L_t}{\partial t} \right\|_{L^2(g)} \to 0$ (so that for any $\delta > 0$, there is a $T >> 0$, so that $\|\partial_t L_t\|_{L^2(g)} < \delta$ for $t > T$); and in particular L_t converges to a self-adjoint, strongly elliptic, semi-definite operator of order $2d$, denoted by $L_\infty = \lim_{t \to \infty} L_t$. Then the exists smooth functions $\eta(t)$ and $\psi(t)$ on $[0, \infty)$, with $\eta(t) \leq 0$ and $\psi(t) \geq 0$ (and vanishing in a neighbourhood of 0, so that in particular $w_\varepsilon(t)$ is smooth) such that given any $\varepsilon > 0$, and any*

$g \in V_{0,w_\varepsilon(t)}(E)$,with $g(t) \perp ker(L_t)$ for each t, and $f_0 \in L^2_k(E)$, there exists a unique $f \in V_{2k,w_\varepsilon(t)}(E)$ so that $\partial_t f \in V_{0,a(t)}(E)$), which solves the initial value problem

$$\begin{aligned} \frac{\partial f(t)}{\partial t} + L_t f(t) &= g(t), \\ f(0) &= f_0. \end{aligned}$$

Furthermore there is a parabolic estimate

$$\begin{aligned} & \|\partial_t f\|^2_{V_{0,w_\varepsilon(t)}(g)} + \|f\|^2_{V_{2k,w_\varepsilon(t)}(g)} \\ \leq \;& C\left(\|f_0\|_{L^2_k(g)} + \|g\|_{V_{0,w_\varepsilon(t)}(g)}\right), \end{aligned}$$

where the constant C depends only on ε,$\eta(t)$, $\psi(t)$ and L_t.

Proof. We will put ourselves in a position to apply the Lax-Lions-Milgram theorem, which requires us in particular to produce a subspace $H \subset V_{k,w_\varepsilon(t)}$, a sesquilinear form $B : V_{k,w_\varepsilon(t)} \times H \to \mathbb{R}$, and a bounded linear functional $F : H \to \mathbb{R}$. To motivate the definitions, notice that if $f \in C_0^\infty(M \times [0,\infty))$ solves the initial value problem

$$\begin{aligned} \partial_t f + L_t f &= g \\ f(0) &= f_0, \end{aligned}$$

where $L(t)$ is for each t a self-adjoint elliptic operator of order $2k$, then integrating by parts and writing $\langle -,- \rangle$ for the L^2 inner product we must have

$$\begin{aligned} \int_0^\infty |w_\varepsilon(t)|^2 \langle g, L\phi\rangle\, dt &= \int_0^\infty |w_\varepsilon(t)|^2 \langle \partial_t f + L_t f, L_t\phi\rangle\, dt = \int_0^\infty |w_\varepsilon(t)|^2 (\langle \partial_t f, L_t\phi\rangle + \langle L_t f, L_t \phi\rangle)dt \\ &= \int_0^\infty |w_\varepsilon(t)|^2 (\langle L_t f, L_t\phi\rangle - \langle f, \partial_t(L_t\phi)\rangle)\, dt - \int_0^\infty 2w_\varepsilon(t)w_\varepsilon^{'}(t)\langle f, L_t\phi\rangle\, dt - \langle f_0, L_0\phi(0)\rangle, \end{aligned}$$

where for the moment we will write the weight function as $w_\varepsilon(t) = e^{-\eta(t)}t^{-\varepsilon\psi(t)}$, where η and ψ will be defined later, with the understanding that the chosen functions will make $w_\varepsilon(t)$ smooth with $w_\varepsilon(0) = 1$. Endow the space $C_0^\infty(M \times [0,\infty))$ with a norm $\|-\|_H$ given by

$$\|\phi\|_H = \|\phi(0)\|^2_{L^2_k(M)} + \|\phi\|^2_{V_{2k,w_\varepsilon(t)}},$$

and write H for the corresponding normed space. Clearly $H \hookrightarrow V_{k,w_\varepsilon(t)}$. Then define the sesquilinear form $B : V_{k,w_\varepsilon(t)} \times H \to \mathbb{R}$ by

$$B(f,\phi) = \int_0^\infty |w_\varepsilon(t)|^2 (\langle L_t f, L_t\phi\rangle - \langle f, \partial_t(L_t\phi)\rangle) - 2w_\varepsilon(t)w_\varepsilon^{'}(t)\langle f, L_t\phi\rangle\, dt,$$

and the linear functional $F : H \to \mathbb{R}$

$$F(\phi) = \langle f_0, L_0\phi(0)\rangle + \int_0^\infty |w_\varepsilon(t)|^2 \langle g, L_t\phi\rangle.$$

Now by Cauchy-Scwartz we have

$$|B(f,\phi)| \leq \int_0^\infty |w_\varepsilon(t)|^2 (\|L_t f\|_{L^2}\|L_t\phi\|_{L^2} + \|f\|_{L^2}\|\partial_t(L_t\phi)\|_{L^2}) + 2|w_\varepsilon(t)|\left|w_\varepsilon^{'}(t)\right|\|f\|_{L^2}\|L\phi\|_{L^2}.$$

Now if $\left|w_\varepsilon^{'}(t)\right| \leq Cw_\varepsilon(t)$, then we have that

$$|B(f,\phi)| \leq C\int_0^\infty |w_\varepsilon(t)|^2 \|f\|^2_{L^2_{2k}}\, dt = \|f\|^2_{V_{2k,w_\varepsilon(t)}},$$

so that the map $W \to \mathbb{R}$ given by $f \to B(f,\phi)$ is bounded. Similarly, integrating by parts and applying Cauchy-Schwartz gives

$$\begin{aligned}
|F(\phi)| &\leq C\left\|f_0\right\|_{L^k}\left\|\phi(0)\right\|_{L^k}+\int_0^\infty |w_\varepsilon(t)|^2\left\|g\right\|_{L^2}\left\|L\phi\right\|_{L^2}dt \\
&\leq C\left(\left\|f_0\right\|_{L^k}\left\|\phi(0)\right\|_{L^k}+\left(\int_0^\infty |w_\varepsilon(t)|^2\left\|g\right\|_{L^2}^2 dt\right)^{1/2}\left(\int_0^\infty |w_\varepsilon(t)|^2\left\|\phi\right\|_{L^2_{2k}}^2 dt\right)^{1/2}\right) \\
&= C\left(\left\|f_0\right\|_{L^k}\left\|\phi(0)\right\|_{L^k}+\left\|g\right\|_{V_{0,w_\varepsilon(t)}}^2\left\|\phi\right\|_{V_{2k,w_\varepsilon(t)}}^2\right) \\
&\leq C\left(\left\|f_0\right\|_{L^k}+\left\|g\right\|_{V_{0,w_\varepsilon(t)}}^2\right)\left(\left\|\phi(0)\right\|_{L^k}+\left\|\phi\right\|_{V_{2k,w_\varepsilon(t)}}\right) \\
&= C\left(\left\|f_0\right\|_{L^k}+\left\|g\right\|_{V_{0,w_\varepsilon(t)}}^2\right)\left\|\phi\right\|_H = C\left\|\phi\right\|_H
\end{aligned}$$

so $F : W \to \mathbb{R}$ is also bounded.In order to apply Lax-Milgram-Lions we need to check coercivity. First notice that, using the fact that L is self-adjoint:

$$\begin{aligned}
\partial_t(|w_\varepsilon(t)|^2\langle\phi,L\phi\rangle) &= 2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+|w_\varepsilon(t)|^2\left(\langle\partial_t\phi,L_t\phi\rangle+\langle\phi,\partial_t\left(L_t\phi\right)\rangle\right) \\
&= 2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+|w_\varepsilon(t)|^2\left(\langle L_t(\partial_t\phi),\phi\rangle+\langle\phi,(\partial_tL_t)(\phi)\rangle+\langle\phi,L_t(\partial_t(\phi)\rangle\right) \\
&= 2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+2\left|w_\varepsilon(t)\right|^2\left(\langle\phi,(\partial_tL_t)(\phi)\rangle+\langle\phi,L_t(\partial_t(\phi))\rangle\right)-|w_\varepsilon(t)|^2\langle\phi,(\partial_tL_t)(\phi)\rangle \\
&= 2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+2\left|w_\varepsilon(t)\right|^2\left(\langle\phi,\partial_t\left(L_t\phi\right)\rangle\right)-|w_\varepsilon(t)|^2\langle\phi,(\partial_tL_t)(\phi)\rangle\,.
\end{aligned}$$

Now

$$\begin{aligned}
-\langle\phi(0),L_0\phi(0)\rangle &= \int_0^\infty \partial_t(|w_\varepsilon(t)|^2\langle\phi,L_t\phi\rangle)dt \\
&= \int_0^\infty 2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+|w_\varepsilon(t)|^2\left(\left(2\langle\phi,\partial_t\left(L_t\phi\right)\rangle\right)-\langle\phi,(\partial_tL_t)(\phi)\rangle\right)dt,
\end{aligned}$$

so

$$\int_0^\infty |w_\varepsilon(t)|^2\langle\phi,\partial_t\left(L\phi\right)\rangle dt = -\frac{1}{2}\langle\phi(0),L\phi(0)\rangle-\int_0^\infty w_\varepsilon(t)w^{'}(t)\langle\phi,L\phi\rangle dt+\frac{1}{2}\int_0^\infty |w_\varepsilon(t)|^2\langle\phi,(\partial_tL)(\phi)\rangle dt.$$

From this it follows that

$$\begin{aligned}
B(\phi,\phi) &= \int_0^\infty |w_\varepsilon(t)|^2\left(\left\|L_t\phi\right\|_{L^2}^2-\langle\phi,\partial_t(L_t\phi)\rangle\right)-2w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle dt \\
&= \int_0^\infty |w_\varepsilon(t)|^2\left(\left\|L_t\phi\right\|_{L^2}^2-\frac{1}{2}\langle\phi,(\partial_tL_t)(\phi)\rangle\right)-w_\varepsilon(t)w^{'}(t)\langle\phi,L_t\phi\rangle+\frac{1}{2}|w(0)|^2\langle\phi(0),L_t\phi(0)\rangle\,.
\end{aligned}$$

If we assume L_t is positive definite for all t, then Gårding's inequality applies to give a constant c such that

$$\langle\phi(0),L_t\phi(0)\rangle \geq c\left\|\phi(0)\right\|_{L^2_k}.$$

and

$$\left\|L_t\phi(t)\right\|_{L^2}^2 \geq c\left\|\phi(t)\right\|_{L^2_{2k}},$$

where for the second inequality we use the fact that $\langle L_t\phi(t),L_t\phi(t)\rangle = \langle L_tL_t\phi(t),\phi(t)\rangle$, and L_t^2 is a positive definite elliptic operator of order $4k$. By Cauchy-Schwarz we have

$$|\langle\phi,L_t\phi(t)\rangle| \leq C\left\|\phi\right\|_{L^2_k}^2 \leq C\left\|\phi\right\|_{L^2_{2k}}^2.$$

Furthermore we may write

$$-\frac{1}{2}\langle\phi,(\partial_tL_t)(\phi)\rangle \geq -\frac{1}{2}\left\|\partial_tL_t\right\|_{L^2}\left\|\phi\right\|_{L^2}^2 \geq -\frac{1}{2}\left\|\partial_tL_t\right\|_{L^2}\left\|\phi\right\|_{L^2_{2k}}^2,$$

where $\|\partial_t L_t\|_{L^2}$ is the operator norm induced by the L^2 norm. Then we obtain

$$\begin{aligned} B(\phi,\phi) &\geq \int_0^\infty |w_\varepsilon(t)|^2 c\,\|\phi(t)\|_{L^2_{2k}} - \frac{1}{2}|w_\varepsilon(t)|^2 \|\partial_t L_t\|_{L^2}\|\phi\|^2_{L^2_{2k}} - Cw_\varepsilon(t)w^{'}(t)\|\phi\|^2_{L^2_{2k}}\,dt + c^{'}\|\phi(0)\|_{L^2_k} \\ &= \int_0^\infty \left(cw_\varepsilon(t) - \frac{1}{2}w_\varepsilon(t)\|\partial_t L_t\|_{L^2} - Cw^{'}_\varepsilon(t)\right) w_\varepsilon(t)\|\phi\|^2_{L^2_{2k}}\,dt + c^{'}\|\phi(0)\|_{L^2_k}, \end{aligned}$$

so we want that

$$cw_\varepsilon(t) - \frac{1}{2}w_\varepsilon(t)\|\partial_t L_t\|_{L^2} - Cw^{'}_\varepsilon(t) \geq c^{"}w_\varepsilon(t)$$

for some constant $c'' > 0$, so that

$$\begin{aligned} B(\phi,\phi) &\geq c^{''}\int_0^\infty |w_\varepsilon(t)|^2\|\phi(t)\|_{L^2_{2k}}\,dt + c^{'}\|\phi(0)\|_{L^2_k} \\ &= \|\phi(0)\|^2_{L^2_k(M)} + \|\phi\|^2_{V_{2k,w_\varepsilon(t)}} = \|\phi\|_W, \end{aligned}$$

establishing coercivity.

Fix $\delta > 0$, small enough so that $c - \delta > 0$ and choose any sufficiently large $T \in [0,\infty)$ so that $\|\partial_t L_t\|_{L^2} \leq \delta$ when $t > T$. We will set $w_\varepsilon(t) = e^{-\eta(t)}t^{-\varepsilon\psi(t)}$, where $\varepsilon > 0$ is an arbitrarily small number and $\eta(t)$ and $\psi(t)$ are smooth functions defined as follows. Let χ be smooth cutoff function which is 1 on $[0,T]$ and is supported in $[0,2T)$. Then define $\eta(t) = a(t-2T)\chi(t)$, where

$$a = \sup_{t\in[0,T]}\frac{1}{2C}\|\partial_t L_t\|_{L^2},$$

and let $\psi(t) = 1-\chi(t)$, so that ψ is 1 on $\left[T^{'},\infty\right)$ and supported on $[T,\infty)$, and so in particular the weight function $w_\varepsilon(t)$ is smooth and equal to $t^{-\varepsilon}$ on $\left[T^{'},\infty\right)$

Then notice that we have

$$\begin{aligned} w_\varepsilon(t)w^{'}(t) &= -e^{-\eta(t)}t^{-\varepsilon\psi(t)}\left(e^{-\eta(t)}t^{-\varepsilon\psi(t)}\eta^{'}(t) + \varepsilon e^{-\eta(t)}t^{-\varepsilon\psi(t)}\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)\right) \\ &= -|w_\varepsilon(t)|^2\left(\eta^{'}(t) + \varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)\right). \end{aligned}$$

Since $\ln(t)$ is only unbounded near 0 and ∞ (where $\psi^{'}(t)$ vanishes), $\psi^{'}(t)\ln(t)$ is clearly bounded. Similarly, since $\frac{1}{t}$ is unbounded only near 0 (where ψ vanishes), $\frac{\psi(t)}{t}$ is also bounded. Therefore

$$\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}$$

is bounded, and therefore so is

$$\eta^{'}(t) + \varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)$$

since $\eta^{'}(t) = a\ \chi(t) + a(t-T^{'})\chi^{'}(t)$, and χ is supported in a bounded interval.

$$\left|w_\varepsilon(t)w^{'}(t)\right| \leq C^{'}|w_\varepsilon(t)|^2,$$

and this gives the boundedness stated above. Then let us analyse the quantity

$$\begin{aligned} & cw_\varepsilon(t) - \frac{1}{2}w_\varepsilon(t)\|\partial_t L\|_{L^2} - Cw^{'}(t) \\ = & \left(c - \frac{1}{2}\|\partial_t L\|_{L^2} + C\left(\eta^{'}(t) + \varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)\right)\right)w_\varepsilon(t). \end{aligned}$$

We need that

$$c - \frac{1}{2}\|\partial_t L\|_{L^2} + C\left(\eta^{'}(t) + \varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)\right) \geq \lambda > 0.$$

for all $t \in [0,\infty)$. For $t \in [0,T]$, by construction this is

$$\begin{aligned} & c - \frac{1}{2}\|\partial_t L\|_{L^2} + C\eta^{'}(t) \\ = \ & c - \frac{1}{2}\|\partial_t L\|_{L^2} + Ca \\ = \ & c - \frac{1}{2}\|\partial_t L\|_{L^2} + \sup_{t\in[0,T]} \frac{1}{2}\|\partial_t L_t\|_{L^2} \geq c > 0. \end{aligned}$$

On the other hand, for $t \in (T,\infty)$ we have that $c - \frac{1}{2}\|\partial_t L\|_{L^2} \geq c - \delta > 0$. We also have

$$\eta^{'}(t) = a\chi(t) + a(t - T^{'})\chi^{'}(t) \geq 0$$

since $a\chi(t)$ is non-negative by construction, $\chi^{'}(t) \leq 0$ and $a(t-T^{'}) \leq 0$ for $t \in \left[0, T^{'}\right]$ and $\chi^{'}(t) \equiv 0$ for $t > T^{'}$. Also note that $\varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right) > 0$ (notice that ψ is positive and increasing), so

$$\begin{aligned} & c - \frac{1}{2}\|\partial_t L\|_{L^2} + C\left(\eta^{'}(t) + \varepsilon\left(\psi^{'}(t)ln(t) + \frac{\psi(t)}{t}\right)\right) \\ = \ & c - \frac{1}{2}\|\partial_t L\|_{L^2} \geq c - \delta > 0. \end{aligned}$$

Then F and $\phi \to B(f,\phi)$ are bounded, and B is coercive, so we apply Lax-Lions-Milgram, so that $F(\phi) = B(f,\phi)$

By the Lax-Lions-Milgram lemma this gives the existence of a $u \in V_{2d,a(t)}(E)$ such that we have $B(u,\phi) = F(\phi)$, for each $\phi \in C_0^\infty\left(M \times [0,\infty), p_M^*(E)\right)$. This means that

$$\int_0^\infty |w(t)|^2 \langle \partial_t f + L_t f, L_t\phi\rangle \, dt = \int_0^\infty |w(t)|^2 \langle g, L_t\phi\rangle$$

where $\partial_t f$ is interpreted in the sense of distributions, with the boundary conditon $f(0) = f_0$. Because L_t is invertible any section of $C_0^\infty\left(M \times [0,\infty), p_M^*(E)\right)$ is equal to $L_t\phi$ for some ϕ, this implies that f is a weak solution to the equation $\partial_t u + L_t u = f$. This also implies that $\partial_t f \in V_{0,w_\varepsilon(t)}(E)$, since $\langle \partial_t u, \phi\rangle_{V_{0,w_\varepsilon(t)}} = \langle f - L_t u, \phi\rangle_{V_{0,w_\varepsilon(t)}}$ for any ϕ.

Finally, the last part of the Lax-Lions-Milgram lemma gives the estimate $\|f\|_{V_{2k,w_\varepsilon(t)}(E)} \leq \frac{c}{\lambda}\|F\| = \frac{c}{\lambda}\sup F(\phi)$, and we have shown

$$|F(\phi)| \leq C\left(\|f_0\|_{L^k} + \|g\|_{V_{0,w_\varepsilon(t)}}\right)\|\phi\|_H$$

so that

$$\|f\|_{V_{2k,w_\varepsilon(t)}(E)} \leq C\left(\|f_0\|_{L^k} + \|g\|_{V_{0,w_\varepsilon(t)}}\right).$$

Since $\partial_t f = g - L_t u$, we have

$$\|\partial_t f\|_{V_{0,w_\varepsilon(t)}} \leq \|g\|_{V_{0,w_\varepsilon(t)}} + \|f\|_{V_{2k,w_\varepsilon(t)}(E)},$$

so that

$$\|\partial_t f\|_{V_{0,w_\varepsilon(t)}} + \|f\|_{V_{2k,w_\varepsilon(t)}(E)} \leq C\left(\|f_0\|_{L^k} + \|g\|_{V_{0,w_\varepsilon(t)}}\right)$$

which is the parabolic estimate in the statement of the theorem. $\square$

7.1.5. *Higher regularity.*

Theorem 7.10. *Let L_t be a family of elliptic operators of order $2d$, converging smoothly as $t \to \infty$ to a self-adjoint semi-definite, strongly elliptic differential operator of order $2k$ denoted by L_∞, so that in particular $||\partial_t L_t||_{L^2} \to 0$. Assume furthermore that $\ker L_t \subset \ker L_\infty$ for all t. Let $K \subset E$ a subspace such that $\ker L_\infty \perp K$ so that in particular $\ker L_t \perp K$. Then there exists a path of smooth functions $\eta(t) \leq 0$ and $\psi(t) \geq 0$ (and vanishing in a neighbourhood of 0) such that the such that for any $\varepsilon > 0$, the associated weight function $w_\varepsilon(t)$ is smooth, and such that given any $p \in \mathbb{N}$, $g(t) \in W_{2d,p,q-1,w_\varepsilon(t)}(E,K)$ and $f_0 \in L^2_{(2d+1)p}$, there exists a unique weak solution $f(t) \in W_{2d,p+1,q,w_\varepsilon(t)}(E,K)$ to the initial value problem*

$$\begin{aligned} \partial_t f(t) + L_t f(t) &= g(t) \\ f(0) &= f_0. \end{aligned} \tag{7.1}$$

There is furthermore a parabolic estimate of the form

$$||f_t||_{W_{2d,p+1,q,w_\varepsilon(t)}} \leq C(||f_0||_{L^2_{(2d+1)p}} + ||g_t||_{W_{2d,p,q-1,w_\varepsilon(t)}}),$$

where C depends only on L_t and the weight functions.

Proof. We will now outline how the inductive argument will work before proceeding with the details. The proof will be by induction on p and q. Note that the case $p = 0$, $q = 1$ is Theorem7.9. Namely, assume the theorem holds for $p-1, q$. In other words, there exists a smooth function $w_\varepsilon(t)$ such that for each $f \in W_{2d,p,q,w_\varepsilon(t)}$ and $f_0 \in L^2_{d(2p+1)}$, so that f and f_0 solve the initial value problem7.1 By assumption L_∞^{-1} exists. Now as before let $g(t) \in W_{2d,p,q-1,w_\varepsilon(t)}(E,K)$ and $u_0 \in L^2_{d(2p+1)}$, and assume for the moment that the system

$$\begin{aligned} \frac{\partial \widetilde{f}_t}{\partial t} + L_\infty L_t L_\infty^{-1} \widetilde{f}_t &= L_\infty g_t \\ \widetilde{f}_t(0) &= L_\infty f_0 \end{aligned}$$

has a solution for $\widetilde{f}(t) \in W_{2d,p,q-1,w_\varepsilon(t)}(E,K)$. Then if we set $f(t) = L_\infty^{-1}\, \widetilde{f}(t)$

$$\begin{aligned} \frac{\partial f_t}{\partial t} &= L_\infty^{-1} \frac{\partial \widetilde{f}_t}{\partial t} = L_\infty^{-1}\left(L_\infty g_t - L_\infty L_t f_t\right) = g_t - L_t f_t, \\ f(0) &= f_0, \end{aligned}$$

so $f(t)$ is the desired solution. Clearly $f(t) \in W_{2d,p+1,q,w_\varepsilon(t)}(E,K) \cap V_{2(p+1)d,w_\varepsilon(t)}$To prove the estimate stated in the theorem note that:

$$\begin{aligned} \|f_t\|^2_{W_{2d,p+1,q,w_\varepsilon(t)}} &= \sum_{j=0}^{q} \left\|\partial_t^j f_t\right\|^2_{V_{2d(p+1-j),w_\varepsilon(t)}} \\ &\leq \|\partial_t^q f_t\|^2_{V_{2d(p+1-q),w_\varepsilon(t)}} + C\sum_{j=0}^{q-1} \left\|\partial_t^j \widetilde{f}_t\right\|^2_{V_{2d(p-j),w_\varepsilon(t)}} \\ &= \|\partial_t^q f_t\|^2_{V_{2d(p+1-q),w_\varepsilon(t)}} + C\left\|\widetilde{f}_t\right\|^2_{W_{2d,p,q-1,w_\varepsilon(t)}} \\ &= \left\|\partial_t^{q-1}\left(g_t - L_t L_\infty^{-1} \widetilde{f}(t)\right)\right\|^2_{V_{2d(p+1-q),w_\varepsilon(t)}} + C\left\|\widetilde{f}_t\right\|^2_{W_{2d,p,q-1w_\varepsilon(t)}} \\ &\leq C\left(\left\|\partial_t^{q-1} g_t\right\|^2_{V_{2d(p+1-q),w_\varepsilon(t)}} + \left\|\widetilde{f}_t\right\|^2_{W_{2d,p,q-1,w_\varepsilon(t)}}\right). \end{aligned}$$

In the second line we have used that $L_\infty f(t) = \widetilde{f}(t)$ so that the $L^2_{2d(p-j)}$ norm of $\partial_t^j \widetilde{f}_t$ controls the $L^2_{2d(p+1-j)}$ norm of $\partial_t^j f_t$. In the last line we have used $\partial_t^q f_t = \partial_t^{q-1}(g_t - L_t f_t)$, used the boundedness of $L_t L_\infty^{-1}$ (and its time derivatives) to bound $||\partial_t^{q-1}\left(L_t L_\infty^{-1}\widetilde{f}(t)\right)||^2_{V_{2d(p+1-q),w_\varepsilon(t)}}$ by a constant times a sum of terms of the form $\left\|\partial_t^j \widetilde{f}_t\right\|^2_{V_{2d(p-j),w_\varepsilon(t)}}$, which we have absorbed into the term $||\widetilde{f}_t||_{W_{2d,p,q-1,w_\varepsilon(t)}}$ where they already appear. Applying the parabolic estimate inductively to $\widetilde{f}_t$ we get

$$\begin{aligned}
\|f_t\|^2_{W_{2d,p+1,q,w_\varepsilon(t)}} &\leq C\left(\left\|\partial_t^{q-1} g_t\right\|^2_{V_{2d(p+1-q),w_\varepsilon(t)}} + \|L_\infty g_t\|^2_{W_{2d,p-1,q-2,w_\varepsilon(t)}} + \|Lu_0\|_{L^2_{2d((p-1)+1))}}\right) \\
&\leq C\left(\left\|\partial_t^{q-1} g_t\right\|^2_{V_{2d(p-(q-1)),w_\varepsilon(t)}} + \|g_t\|^2_{W_{2d,p,q-2,w_\varepsilon(t)}} + \|u_0\|_{L^2_{2d(p+1)}}\right) \\
&= C\left(\|g_t\|^2_{W_{2d,p,q-1,w_\varepsilon(t)}} + \|u_0\|_{L^2_{2d(p+1)}}\right).
\end{aligned}$$

Note that this estimate also shows uniqueness, since if we apply it to the difference of two solutions, the right hand side is 0, and therefore the two solutions must be equal.

□

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